On a conjecture by Pierre Cartier about a group of associators

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Abstract

In [8], Pierre Cartier conjectured that for any non commutative formal power series Φ on $X = \{x_0, x_1\}$ with coefficients in a $\mathbb{Q}$-extension, A, subjected to some suitable conditions, there exists an unique algebra homomorphism φ from the $\mathbb{Q}$-algebra generated by the convergent polyzêtas to A such that Φ is computed from Φ_{KZ} Drinfel'd associator by applying φ to each coefficient. We prove φ exists and it is a free Lie exponential over X. Moreover, we give a complete description of the kernel of polyzêta and draw some consequences about a structure of the algebra of convergent polyzêtas and about the arithmetical nature of the Euler constant.

Contents

1 Introduction

1.1 Drinfel'd associator and polyzêtas

In 1986, in order to study the linear representation of the braid group B_n coming from the monodromy of the Knizhnik-Zamolodchikov differential equations over $\mathbb{C}_*^n = \{\underline{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}$ [14] :

$$dF(\underline{z}) = \Omega_n(\underline{z})F(\underline{z}) \quad \text{with} \quad \Omega_n(\underline{z}) = \frac{1}{2\mathrm{i}\pi} \sum_{1 \le i < j \le n} t_{i,j} \frac{d(z_i - z_j)}{z_i - z_j}, \tag{1}$$

and $\{t_{i,j}\}_{i,j \ge 1}$ are noncommutative variables, Drinfel'd introduced a class of formal power series Φ on noncommutative variables over the finite alphabet $X = \{x_0, x_1\}$. Such a power series Φ is called an *associator*.

Since the system (1) is completely integrable then $d\Omega_n - \Omega_n \wedge \Omega_n = 0$ [7, 14]. It is equivalent to the fact the $\{t_{i,j}\}_{i,j\geq 1}$ satisfy the infinitesimal braid relations :

$$\begin{aligned}
t_{i,j} &= 0 &&\text{for} && i = j, &(2)\\
t_{i,j} &= t_{j,i} &&\text{for} && i \neq j, &(3)\\
[t_{i,j}, t_{i,k} + t_{j,k}] &= 0 &&\text{for distinct} && i,j,k, &(4)\\
[t_{i,j}, t_{k,l}] &= 0 &&\text{for distinct} && i,j,k,l. &(5)
\end{aligned}$$

Example 1. • $\mathcal{T}_2 = \{t_{1,2}\}$.

$$\Omega_2(z_1, z_2) = \frac{t_{1,2}}{2i\pi}\frac{d(z_1 - z_2)}{z_1 - z_2} \quad \textit{with} \quad F(z_1, z_2) = (z_1 - z_2)^{t_{1,2}/2i\pi}.$$

• $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$, $[t_{1,3}, t_{1,2} + t_{2,3}] = [t_{2,3}, t_{1,2} + t_{1,3}] = 0$.

$$\begin{aligned}
\Omega_3(z_1, z_2, z_3) &= \frac{1}{2i\pi}\left[t_{1,2}\frac{d(z_1 - z_2)}{z_1 - z_2} + t_{1,3}\frac{d(z_1 - z_3)}{z_1 - z_3} + t_{2,3}\frac{d(z_2 - z_3)}{z_2 - z_3}\right].\\
F(z_1, z_2, z_3) &= G\left(\frac{z_1 - z_2}{z_1 - z_3}\right)(z_1 - z_3)^{(t_{1,2}+t_{1,3}+t_{2,3})/2i\pi},
\end{aligned}$$

where G *satisfies the following fuchsian differential equation with three regular singularities at* $0, 1$ *and* ∞ *:*

$$(DE) \qquad dG(z) = [x_0\, \omega_0(z) + x_1\, \omega_1(z)]G(z),$$

with

$$\begin{aligned}
x_0 &:= \frac{t_{1,2}}{2i\pi} \quad \textit{and} \quad \omega_0(z) := \frac{dz}{z},\\
x_1 &:= -\frac{t_{2,3}}{2i\pi} \quad \textit{and} \quad \omega_1(z) := \frac{dz}{1-z}.
\end{aligned}$$

As already shown by Drinfel'd, the equation (DE) admits, on the simply connected domain $\mathbb{C} - (]-\infty, 0] \cup [1, +\infty[)$, two specific solutions

$$G_0(z) \widetilde{_{z\rightsquigarrow 0}} \exp[x_0 \log(z)] \quad \text{and} \quad G_1(z) \widetilde{_{z\rightsquigarrow 1}} \exp[-x_1 \log(1-z)]. \tag{6}$$

He also proved there exists the associator Φ_{KZ} such that $G_1^{-1}(z)G_0(z) = \Phi_{KZ}$.

After that, Lê and Murakami expressed the coefficients of the Drinfel'd associator Φ_{KZ} in terms of *convergent* polyzêtas [22], *i.e.* for $r_1 > 1$,

$$\zeta(r_1, \ldots, r_k) = \sum_{n_1 > \ldots > n_k > 0} \frac{1}{n_1^{r_1} \ldots n_k^{r_k}}. \tag{7}$$

In [22], the authors also expressed the *divergent* coefficients as *linear* combinations of convergent polyzêtas via a *regularization process* (see also [38]). This process is one of many ways to regularize the divergent terms.

1.2 Group of associators and regularized Chen generating series

The algebraic aspects of our regularization process based essentially on various products[1] among polyzêtas (see [33]) and its analytical aspects will be described, in Section 3.1, as the *finite part*, of the asymptotic expansions in different scales of comparison[2] [5]. It will be seen also, in Section 3.2, as the action of the differential Galois group of the polylogarithms[3] (recalled in Section 2.1.2)

$$\mathrm{Li}_{r_1,\ldots,r_k}(z) \quad = \sum_{n_1>\ldots>n_k>0} \frac{z^{n_1}}{n_1^{r_1}\ldots n_k^{r_k}} \tag{8}$$

on the asymptotic expansion of polylogarithms, at $z=1$ and in the comparison scale $\{(1-z)^a \log^b(1-z)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$, and the same action on the asymptotic expansions, at $+\infty$ and in the comparison scales $\{n^a \log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$ and $\{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$, of the harmonic sums (recalled in Section 2.1.1)

$$\mathrm{H}_{r_1,\ldots,r_k}(N) \quad = \sum_{n_1>\ldots>n_k>0}^{N} \frac{1}{n_1^{r_1}\ldots n_k^{r_k}}. \tag{9}$$

This action leads then to a conjecture by Pierre Cartier ([8], conjecture C3) and to the description of the group of associators yielding the ideal of polynomial relations among coefficients of associators (theorems 13 and 14). This group is in fact, closely linked to the group of the Chen generating series studied by K.T. Chen to describe the solutions of differential equations [10] and it turns out that each associator regularizes a Chen generating series of the differential forms ω_0 and ω_1 along the integration path on the simply connected domain $\mathbb{C}-(]-\infty,0]\cup[1,+\infty[)$.

1.3 Global renormalization and global regularization

In fact, our regularization process based essentially on two noncommutative generating series over $Y=\{y_i\}_{i\geq 1}$, which encodes the multi-indices $(r_1,\ldots,r_k)$ by the words $y_{r_1}\ldots y_{r_k}$ over the monoid generated by Y, denoted by Y^*, of polylogarithms and of harmonic sums (recalled in Section 2.2.1)

$$\Lambda(z) = \sum_{w\in Y^*} \mathrm{Li}_w(z)\, w \quad \text{and} \quad \mathrm{H}(N) = \sum_{w\in Y^*} \mathrm{H}_w(N)\, w. \tag{10}$$

Through the algebraic combinatorial aspects[4] [45] and the topological aspects [2] of formal power series in noncommutative variables, we have already

[1] First source of ambiguity leading to the problem of rewriting expressions of polyzêtas in a canonical form using irreducible Lyndon words (see [34, 32]).

[2] Second source of ambiguity leading to the problem to determine the value of regularized polyzêtas and its analytical meaning (see [33, 37]).

[3] Third source of ambiguity leading to the problem of fixing the integration path to solve (DE) and its monodromy group (see [35]) or its differential Galois group (see [38]).

[4] See [45] to get an idea of these aspects of combinatorial Hopf algebra of the shuffle product,

showed the existence of noncommutative formal series over Y, Z_1 and Z_2 with constant terms, such that [39]

$$\lim_{z\to 1} \exp\left(y_1 \log \frac{1}{1-z}\right)\Lambda(z) = Z_1, \tag{11}$$

$$\lim_{N\to\infty} \exp\left(\sum_{k\geq 1} \mathrm{H}_{y_k}(N)\frac{(-y_1)^k}{k}\right)\mathrm{H}(N) = Z_2. \tag{12}$$

Moreover, Z_1 and Z_2 are equal and stand for the noncommutative generating series of all convergent polyzêtas $\{\zeta(w)\}_{w\in Y^*-y_1Y^*}$ as shown by the factorized form indexed by Lyndon words (recalled in Section 2.2). This theorem enables, in particular, to explicit the counter-terms eliminating the divergence of the polylogarithms $\{\mathrm{Li}_w(z)\}_{w\in y_1Y^*}$, for $z\to 1$, and of the harmonic sums $\{\mathrm{H}_w(N)\}_{w\in y_1Y^*}$, for $N\to\infty$, and to calculate the Euler-Mac Laurin constants associated to the divergent polyzêtas $\{\zeta(w)\}_{w\in y_1Y^*}$ (see Corollary 4). It allows also to give, in Section 3.3 and via identification of locale coordinates in infinite dimension, a *complete* description of the kernel by its generators, of the following algebra homomorphism[5]

$$\zeta : (A\epsilon \oplus (Y-y_1)A\langle Y\rangle, \uplus) \longrightarrow (\mathbb{R}, .) \tag{13}$$

$$y_{r_1}\ldots y_{r_k} \longmapsto \sum_{n_1>\ldots>n_k>0} \frac{1}{n_1^{r_1}\ldots n_k^{r_k}}, \tag{14}$$

and the set of A-*irreducible* polyzêtas forming a transcendence basis of the image of ζ, with $A=\mathbb{Q}[\mathrm{i}\pi]$ (see Corollary 10).

Finally, via the *indiscernability* (recalled in Section 2.3) over the group of associators, this study makes precise the structure of the A-algebra generated by the convergent polyzêtas (see Theorem 19) and concludes the main challenge of the *polynomial* relations among polyzêtas indexed by convergent Lyndon words which are algebraicly independant on the Euler constant and motivated [32, 34, 3, 49]. In particular, the A-algebra generated by the convergent polyzêtas was conjectured to be *free* [32, 34] and it will be proved, thanks to the propositions 15, 16 and 17. Moreover, this free A-algebra is *graded by weight* meaning there is no *linear* relation among convergent polyzêtas of different weight (see Theorem 19).

denoted by $\sqcup\!\sqcup$, and its co-product, denoted by $\Delta_{\sqcup\!\sqcup}$. For the quasi-shuffle product, denoted by $\uplus$, and its co-product, denoted by $\Delta_{\uplus}$, see Annexe A.

In our works, recalled in Annexe B, these algebraic combinatorial aspects were explored systematically to expand the outputs of nonlinear controlled dynamical system with singular inputs (Corollary 17) on polylogarithmic functional basis [26, 27, 31]. In this way [39], polyzêtas do appear then as fundamental arithmetical constant for the asymptotic analysis and for the renormalization of the outputs and their successive derivations (Corollary 18) via the extended Fliess fundamental formula (Theorem 23).

[5] Here, ϵ stands for the empty word over Y.

2 Background : structures and analytical studies of harmonic sums and of polylogarithms

2.1 Structures of harmonic sums and of polylogarithms

2.1.1 Quasi-symmetric functions and harmonic sums

Let $\{t_i\}_{i\in\mathbb{N}_+}$ be an infinite set of variables. The elementary symmetric functions η_k and the power sums ψ_k are defined by (see [45])

$$\eta_k(\underline{t}) = \sum_{n_1>\ldots>n_k>0} t_{n_1}\ldots t_{n_k} \quad \text{and} \quad \psi_k(\underline{t}) = \sum_{n>0} t_n^k. \tag{15}$$

They are respectively coefficients of the following generating functions

$$\eta(\underline{t}\mid z) = \prod_{i\geq 1}(1+t_i z) \quad \text{and} \quad \psi(\underline{t}\mid z) = \sum_{i\geq 1}\frac{t_i z}{1-t_i z}. \tag{16}$$

These generating functions satisfy a Newton identity

$$z\frac{d}{dz}\log\eta(\underline{t}\mid z) \quad = \quad \psi(\underline{t}\mid -z). \tag{17}$$

The fundamental theorem from symmetric functions theory asserts that $\{\eta_k\}_{k\geq 0}$ are linearly independent, and provides remarkable identities like (with $\eta_0 = 1$) :

$$\eta_k \quad = \quad \frac{(-1)^k}{k!} \sum_{\substack{s_1,\ldots,s_k\geq 0\\ s_1+\ldots+ks_k=k+1}} \binom{k}{s_1,\ldots,s_k}\left(-\frac{\psi_1}{1}\right)^{s_1}\ldots\left(-\frac{\psi_k}{k}\right)^{s_k}. \tag{18}$$

Let Y be the infinite alphabet $\{y_i\}_{i\geq 1}$ equipped with the order $y_1 > y_2 > y_3 > \ldots$ and let $\mathcal{L}ynY$ be the set of Lyndon words over Y. The length of $w = y_{s_1}\ldots y_{s_r} \in Y^*$ is denoted by $\mid w \mid$ and its degree equals to $s_1+\ldots+s_r$.

The quasi-symmetric function F_w, of depth $r = \mid w \mid$ and of degree (or weight) $s_1+\ldots+s_r$, is defined by

$$\mathrm{F}_w(\underline{t}) \quad = \sum_{n_1>\ldots>n_r>0} t_{n_1}^{s_1}\ldots t_{n_r}^{s_r}. \tag{19}$$

In particular, $\mathrm{F}_{y_1^k} = \eta_k$ and $\mathrm{F}_{y_k} = \psi_k$. The functions $\{\mathrm{F}_{y_1^k}\}_{k\geq 0}$ are linearly independent and integrating differential equation (17) shows that functions $\mathrm{F}_{y_1^k}$ and F_{y_k} are linked by the formula

$$\sum_{k\geq 0}\mathrm{F}_{y_1^k} z^k \quad = \quad \exp\left(-\sum_{k\geq 1}\mathrm{F}_{y_k}\frac{(-z)^k}{k}\right). \tag{20}$$

Every $\mathrm{H}_w(N)$ can be obtained by specializing, in the quasi-symmetric function F_w, the variables $\{t_i\}_{i\geq 1}$ as follows [41]

$$\forall N \geq i \geq 1, t_i = 1/i \quad \text{and} \quad \forall i > N, t_i = 0. \tag{21}$$

In the same way, for $w \in Y^* - y_1Y^*$, the convergent polyzêta $\zeta(w)$ can be obtained by specializing, in F_w, the variables $\{t_i\}_{i\geq 1}$ as follows [41]

$$\forall N \geq i \geq 1, \qquad t_i = 1/i. \tag{22}$$

The notation F_w is extended by linearity to all polynomials over Y.

If $u, v \in Y^*$, of length r, s and of weight[6] p, q respectively, $\mathrm{F}_{u \uplus v}$ is a quasi-symmetric function of depth $r + s$ and of weight $p + q$, and $\mathrm{F}_{u \uplus v} = \mathrm{F}_u\ \mathrm{F}_v$, where $\uplus$ is the quasi-shuffle product[7] [41]. Hence,

$$\forall u, v \in Y^*, \qquad \mathrm{H}_{u \uplus v} = \mathrm{H}_u\ \mathrm{H}_v, \tag{23}$$

$$\Rightarrow \quad \forall u, v \in Y^* - y_1Y^*, \qquad \zeta(u \uplus v) = \zeta(u)\ \zeta(v). \tag{24}$$

Remarkable identity (18) can be then seen as

$$y_1^k \quad = \quad \frac{(-1)^k}{k!} \sum_{\substack{s_1,\ldots,s_k\geq 0\\ s_1+\ldots+ks_k=k+1}} \binom{k}{s_1,\ldots,s_k} \frac{(-y_1)^{\uplus s_1}}{1^{s_1}} \uplus \ldots \uplus \frac{(-y_k)^{\uplus s_k}}{k^{s_k}} \tag{25}$$

2.1.2 Iterated integrals and polylogarithms

Let X be the finite alphabet $\{x_0, x_1\}$ equipped with the order $x_0 < x_1$[8]. Let

$$\mathcal{C} := \mathbb{C}\left[z, \frac{1}{z}, \frac{1}{1-z}\right] \quad \text{and} \quad \mathcal{G} := \left\{z, \frac{1}{z}, \frac{z-1}{z}, \frac{z}{z-1}, \frac{1}{1-z}, 1-z\right\}. \tag{26}$$

This ring $\mathcal{C}$ is invariant under differentiation and under the homographic transformations belonging to the group $\mathcal{G}$ whose elements commute the singularities $\{0, 1, +\infty\}$.

The iterated integral over ω_0, ω_1 associated to the word $w = x_{i_1} \cdots x_{i_k}$ over X^* (the monoid generated by X) and along the integration path $z_0 \rightsquigarrow z$ is the following multiple integral defined by

$$\int\limits_{z_0 \rightsquigarrow z} \omega_{i_1} \cdots \omega_{i_k} = \int_{z_0}^{z} \omega_{i_1}(t_1) \int_{z_0}^{t_1} \omega_{i_2}(t_2) \ldots \int_{z_0}^{t_{r-2}} \omega_{i_r}(t_{r-1}) \int_{z_0}^{t_{r-1}} \omega_{i_r}(t_r), \tag{27}$$

where $t_1 \cdots t_{r-1}$ is a subdivision of the path $z_0 \rightsquigarrow z$. In a shortened notation, we denote this integral by $\alpha_{z_0}^z(w)$ and[9] $\alpha_{z_0}^z(\epsilon) = 1$. One can check that the polylogarithm $\mathrm{Li}_{s_1,\ldots,s_r}$ is also the value of the iterated integral over ω_0, ω_1 and along the integration path $0 \rightsquigarrow z$ [29, 31] :

$$\mathrm{Li}_w(z) \quad = \quad \alpha_0^z(x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1). \tag{28}$$

The definition of polylogarithms is extended over the words $w \in X^*$ by putting $\mathrm{Li}_{x_0}(z) := \log z$. The $\{\mathrm{Li}_w\}_{w\in X^*}$ are $\mathcal{C}$-linearly independent [35, 32]. The

[6] The weight is as in Equation (19).
[7] See Annexe A, for the study of the Hopf algebra of $\uplus$ which is not included in [45].
[8] In all the sequel, we follow the notations of [2, 45].
[9] Here, ϵ stands for the empty word over X.

functions $\mathrm{P}_w(z) := (1-z)^{-1}\,\mathrm{Li}_w(z), w \in X^*$, are also $\mathbb{C}$-linearly independent. Since, for $w \in Y^*, \mathrm{P}_w$ is the ordinary generating function of $\{\mathrm{H}_w(N)\}_{N\geq 0}$ [37] :

$$\mathrm{P}_w(z) \quad = \quad \sum_{N\geq 0} \mathrm{H}_w(N)\; z^N \tag{29}$$

then, as a consequence of the classical isomorphism between convergent Taylor series and their associated sums, the harmonic sums $\{\mathrm{H}_w\}_{w\in Y^*}$ are also $\mathbb{C}$-linearly independent. Firstly, $\ker \mathrm{P} = \{0\}$ and $\ker \mathrm{H} = \{0\}$, and secondly, P is a morphism for the Hadamard product :

$$\mathrm{P}_u(z) \odot \mathrm{P}_v(z) = \sum_{N\geq 0} \mathrm{H}_u(N)\mathrm{H}_v(N)z^N = \sum_{N\geq 0} \mathrm{H}_{u \uplus v}(N)z^N = \mathrm{P}_{u \uplus v}(z). \tag{30}$$

Proposition 1 ([37]). *Extended by linearity, the following maps are isomorphism of algebras*

$$\begin{array}{rcl}
\mathrm{P} : (\mathbb{C}\langle Y\rangle, \uplus\,) & \longrightarrow & (\mathbb{C}\{\mathrm{P}_w\}_{w\in Y^*}, \odot)\,, \\
u & \longmapsto & \mathrm{P}_u, \\
\mathrm{H} : (\mathbb{C}\langle Y\rangle, \uplus\,) & \longrightarrow & (\mathbb{C}\{\mathrm{H}_w\}_{w\in Y^*}, .)\,, \\
u & \longmapsto & \mathrm{H}_u = \{\mathrm{H}_u(N)\}_{N\geq 0}.
\end{array}$$

Studying the equivalence between action of $\{(1-z)^l\}_{l\in\mathbb{Z}}$ over $\{\mathrm{P}_w(z)\}_{w\in Y^*}$ and that of $\{N^k\}_{k\in\mathbb{Z}}$ over $\{\mathrm{H}_w(N)\}_{w\in Y^*}$ (see [12]), we have

Theorem 1 ([39]). *The Hadamard $\mathcal{C}$-algebra of $\{\mathrm{P}_w\}_{w\in Y^*}$ can be identified with that of $\{\mathrm{P}_l\}_{l\in\mathcal{L}ynY}$. In the same way, the algebra of harmonic sums $\{\mathrm{H}_w\}_{w\in Y^*}$ with polynomial coefficients can be identified with that of $\{\mathrm{H}_l\}_{l\in\mathcal{L}ynY}$.*

By Identity (25) and by applying the isomorphism H on the set of Lyndon words $\{y_r\}_{1\leq r\leq k}$, we obtain $\mathrm{H}_{y_1^k}$ as polynomials in $\{\mathrm{H}_{y_r}\}_{1\leq r\leq k}$ (which are algebraically independent), and

$$\mathrm{H}_{y_1^k} \quad = \quad \sum_{\substack{s_1,\ldots,s_k\geq 0\\ s_1+\ldots+ks_k=k+1}} \frac{(-1)^k}{s_1!\ldots s_k!}\left(-\frac{\mathrm{H}_{y_1}}{1}\right)^{s_1}\ldots\left(-\frac{\mathrm{H}_{y_k}}{k}\right)^{s_k}. \tag{31}$$

2.2 Results *à la Abel* for generating series of harmonic sums and of polylogarithms

2.2.1 Generating series of harmonic sums and of polylogarithms

Let $\mathrm{H}(N)$ be the noncommutative generating series of $\{\mathrm{H}_w(N)\}_{w\in Y^*}$ [37] :

$$\mathrm{H}(N) \quad := \quad \sum_{w\in Y^*} \mathrm{H}_w(N)\; w. \tag{32}$$

Let $\{\Sigma_w\}_{w\in Y^*}, \{\check{\Sigma}_w\}_{w\in Y^*}$ be respectively a PBW basis of the envelopping algebra $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle Y\rangle)$ and the quasi-shuffle algebra $(\mathbb{Q}\langle Y\rangle, \uplus\,)$ (viewed as a $\mathbb{Q}$-module) on duality such that $\{\Sigma_l\}_{l\in\mathcal{L}ynX}, \{\check{\Sigma}_l\}_{l\in\mathcal{L}ynX}$ are respectively a basis of $\mathcal{L}ie_{\mathbb{Q}}\langle Y\rangle$ and a transcendence basis of $(\mathbb{Q}\langle Y\rangle, \uplus\,)$ (see Annexe A).

Theorem 2 (Factorization of H)**.** *Let*

$$\mathrm{H}_{\mathrm{reg}}(N) \quad := \quad \prod_{l\in\mathcal{L}ynY-\{y_1\}}^{\searrow} e^{\mathrm{H}_{\Sigma_l}(N)\ \Sigma_l}.$$

Then $\mathrm{H}(N) = e^{\mathrm{H}_{y_1}(N)\ y_1}\mathrm{H}_{\mathrm{reg}}(N)$.

Proof. See Annexe A. □

For $l \in \mathcal{L}ynY - \{y_1\}$, the polynomial Σ_l is a finite combination of words in $Y^* - y_1Y^*$. Then we can state the following

Definition 1. *We set* $Z_{⧢} := \mathrm{H}_{\mathrm{reg}}(\infty)$.

The noncommutative generating series of polylogarithms [35, 32]

$$\mathrm{L} \quad := \quad \sum_{w\in X^*} \mathrm{Li}_w\ w \tag{33}$$

satisfies Drinfel'd's differential equation (DE) of Example 1

$$d\mathrm{L} \quad = \quad (x_0\omega_0 + x_1\omega_1)\mathrm{L} \tag{34}$$

with boundary condition [15, 16]

$$\mathrm{L}(\varepsilon) \quad \widetilde{_{\varepsilon\to 0^+}} \quad e^{x_0\log\varepsilon}. \tag{35}$$

This enables us to prove that L is the exponential of a Lie series[10] [35, 32]. Hence,

Proposition 2 (Logarithm of L, [38])**.** *Let* $\pi_1(w)$ *is the following Lie series*

$$\pi_1(w) \quad = \quad \sum_{k\ge 1}\frac{(-1)^{k-1}}{k}\sum_{u_1,\ldots,u_k\in X^+}\langle w \mid u_1 ⧢ \ldots ⧢ u_k\rangle\ u_1\ldots u_k.$$

Then

$$\begin{aligned}\log\mathrm{L}(z) \quad &= \quad \sum_{k\ge 1}\frac{(-1)^{k-1}}{k}\sum_{u_1,\ldots,u_k\in X^+}\mathrm{Li}_{u_1 ⧢ \ldots ⧢ u_k}(z)\ u_1\ldots u_k \\ &= \quad \sum_{w\in X^*}\mathrm{Li}_w(z)\ \pi_1(w).\end{aligned}$$

Applying a theorem of Ree [44, 45], L satisfies Friedrichs criterion [35, 32] :

$$\forall u,v\in X^*, \qquad \mathrm{Li}_{u⧢v} = \mathrm{Li}_u\ \mathrm{Li}_v, \tag{36}$$

$$\Rightarrow \quad \forall u,v\in x_0X^*x_1, \qquad \zeta(u⧢v) = \zeta(u)\ \zeta(v). \tag{37}$$

[10] *i.e.*, L is goup-like for the co-product $\Delta_{⧢}$: $\Delta_{⧢}(\mathrm{L}) = \mathrm{L}\otimes\mathrm{L}$.

Proposition 3 (Successive differentiation of L, [38]). *For any $l \in \mathbb{N}$, let*

$$P_l(z) = \sum_{\mathrm{wgt}(\mathbf{r})=l} \sum_{w \in X^{\deg(\mathbf{r})}} \prod_{i=1}^{\deg(\mathbf{r})} \binom{\sum_{j=1}^{i} r_i + j - 1}{r_i} \tau_{\mathbf{r}}(w) \in \mathcal{C}\langle X \rangle,$$

where, for any $w = x_{i_1} \cdots x_{i_k}$ and $\mathbf{r} = (r_1, \ldots, r_k)$ of degree $\deg(\mathbf{r}) = k$ and of weight $\mathrm{wgt}(\mathbf{r}) = k + r_1 + \cdots + r_k$, the polynomial $\tau_{\mathbf{r}}(w) = \tau_{r_1}(x_{i_1}) \cdots \tau_{r_k}(x_{i_k})$ is defined by

$$\forall r \in \mathbb{N}, \quad \tau_r(x_0) = \partial^r \frac{x_0}{z} = \frac{-r!x_0}{(-z)^{r+1}} \quad \text{and} \quad \tau_r(x_1) = \partial^r \frac{x_1}{1-z} = \frac{r!x_1}{(1-z)^{r+1}}.$$

Denoting $\partial = d/dz$, we have $\partial^l \mathrm{L}(z) = P_l(z)\mathrm{L}(z)$.

Let $\{\check{S}_l\}_{l \in \mathcal{L}ynX}$ be the transcendence basis of the shuffle algebra $(\mathbb{Q}\langle X \rangle, ш)$ and $\{\check{S}_w\}_{w \in X^*}$ be the associated completed basis of the shuffle algebra $(\mathbb{Q}\langle X \rangle, ш)$ (viewed as a $\mathbb{Q}$-module). They are defined as follows [45]

$$\check{S}_{1_{X^*}} = 1 \quad \text{for } l = 1_{X^*} \tag{38}$$

$$\check{S}_l = x\check{S}_u, \quad \text{for } l = xu \in \mathcal{L}ynX, \tag{39}$$

$$\check{S}_w = \frac{\check{S}_{l_1}^{ш\, i_1} ш \ldots ш\, \check{S}_{l_k}^{ш\, i_k}}{i_1! \ldots i_k!} \quad \text{for } w = l_1^{i_1} \ldots l_k^{i_k}, l_1 > \ldots > l_k. \tag{40}$$

Let $\{S_w\}_{w \in Y^*}$ be the PBW basis of the envelopping algebra $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle)$ in duality with the basis $\{\check{S}_w\}_{w \in Y^*}$ and $\{S_l\}_{l \in \mathcal{L}ynX}$ is then the basis of the Lie algebra $\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle$ [45].

Theorem 3 (Factorization of L, [35, 32]). *Let*

$$\mathrm{L}_{\mathrm{reg}} := \prod_{l \in \mathcal{L}ynX - X}^{\searrow} e^{\mathrm{Li}_{S_l} \check{S}_l}.$$

Then $\mathrm{L}(z) = e^{-x_1 \log(1-z)} \mathrm{L}_{\mathrm{reg}}(z) e^{x_0 \log z}$.

For $l \in \mathcal{L}ynX - X$, the polynomial S_l is a finite combination of words in $x_0 X^* x_1$. Then we can state the following

Definition 2 ([35, 32]). *We set $Z_ш := \mathrm{L}_{\mathrm{reg}}(1)$.*

In the definitions 1 and 2 only *convergent* polyzêtas arise and these noncommutative generating series will induce, in Section 3.1, two algebra morphisms of regularization as shown in the theorems 8 and 9 respectively. Hence, these power series are quite different of those given in [22] or in [43] (the last is based on [4], see [8]) needing a regularization process.

2.2.2 Asymptotic expansions by noncommutative generating series and regularized Chen generating series

Let $\rho_{1-z}, \rho_{1-\frac{1}{z}}$ and $\rho_{\frac{1}{z}}$ [36, 32] be three monoid morphisms verifying

$$\rho_{1-z}(x_0) = -x_1 \quad \text{and} \quad \rho_{1-z}(x_1) = -x_0, \tag{41}$$

$$\rho_{1-1/z}(x_0) = -x_0 + x_1 \quad \text{and} \quad \rho_{1-1/z}(x_1) = -x_0 \tag{42}$$

$$\rho_{1/z}(x_0) = -x_0 + x_1 \quad \text{and} \quad \rho_{1/z}(x_1) = x_1. \tag{43}$$

Using homographic transformations belonging to the group $\mathcal{G}$, one has [36, 32]

$$\mathrm{L}(1-z) = e^{x_0 \log(1-z)} \rho_{1-z}[\mathrm{L}_{\mathrm{reg}}(z)] e^{-x_1 \log z} Z_{\sqcup\!\sqcup}, \tag{44}$$

$$\mathrm{L}(1-1/z) = e^{x_0 \log(1-z)} \rho_{1-\frac{1}{z}}[\mathrm{L}_{\mathrm{reg}}(z)] e^{-x_1 \log z} \rho_{1-1/z}(Z_{\sqcup\!\sqcup}^{-1}) e^{\mathrm{i}\pi x_0} \tag{45}$$

$$\mathrm{L}(1/z) = e^{-x_1 \log(1-z)} \rho_{1/z}[\mathrm{L}_{\mathrm{reg}}(z)] e^{(-x_0+x_1)\log z} \rho_{1/z}(Z_{\sqcup\!\sqcup}^{-1}) e^{\mathrm{i}\pi x_1} Z_{\sqcup\!\sqcup}. \tag{46}$$

Thus, (35) and (44) yield [36, 32]

$$\mathrm{L}(z) \underset{z\to 0}{\widetilde{\quad}} \exp(x_0 \log z) \quad \text{and} \quad \mathrm{L}(z) \underset{z\to 1}{\widetilde{\quad}} \exp(-x_1 \log(1-z))\, Z_{\sqcup\!\sqcup}. \tag{47}$$

Let us call $\mathrm{LI}_{\mathcal{C}}$ the smallest algebra containing $\mathcal{C}$, closed under derivation and under integration with respect to ω_0 and ω_1. It is the $\mathcal{C}$-module generated by the polylogarithms $\{\mathrm{Li}_w\}_{w\in X^*}$.

Let $\pi_Y : \mathrm{LI}_{\mathcal{C}}\langle\langle X\rangle\rangle \longrightarrow \mathrm{LI}_{\mathcal{C}}\langle\langle Y\rangle\rangle$ be a projector such that for any $f \in \mathrm{LI}_{\mathcal{C}}$ and $w \in X^*$, $\pi_Y(f\ w x_0) = 0$. Then [39]

$$\Lambda(z) = \pi_Y \mathrm{L}(z) \underset{z\to 1}{\widetilde{\quad}} \exp\left(y_1 \log \frac{1}{1-z}\right) \pi_Y Z_{\sqcup\!\sqcup}. \tag{48}$$

Since the coefficient of z^N in the ordinary Taylor expansion of $\mathrm{P}_{y_1^k}$ is $\mathrm{H}_{y_1^k}(N)$ then let

$$\mathrm{Mono}(z) := e^{-(x_1+1)\log(1-z)} = \sum_{k\geq 0} \mathrm{P}_{y_1^k}(z)\ y_1^k \tag{49}$$

$$\mathrm{Const} := \sum_{k\geq 0} \mathrm{H}_{y_1^k}\ y_1^k = \exp\left(-\sum_{k\geq 1} \mathrm{H}_{y_k} \frac{(-y_1)^k}{k}\right). \tag{50}$$

Proposition 4 ([39])**.** *We have*

$$\pi_Y \mathrm{P}(z) \underset{z\to 1}{\widetilde{\quad}} \mathrm{Mono}(z) \pi_Y Z_{\sqcup\!\sqcup} \quad \textit{and} \quad \mathrm{H}(N) \underset{N\to\infty}{\widetilde{\quad}} \mathrm{Const}(N) \pi_Y Z_{\sqcup\!\sqcup}.$$

Proof. Let μ be the morphism verifying $\mu(x_0) = x_1$ and $\mu(x_1) = x_0$. Then, by Theorem 3, the noncommutative generating series of $\{\mathrm{P}_w\}_{w\in X^*}$ is given by

$$\begin{aligned} \mathrm{P}(z) &= (1-z)^{-1}\mathrm{L}(z) \\ &= e^{-(x_1+1)\log(1-z)} \mathrm{L}_{\mathrm{reg}}(z) e^{x_0 \log z} \\ &= e^{x_0 \log z} \mu[\mathrm{L}_{\mathrm{reg}}(1-z)] e^{-(x_1+1)\log(1-z)} Z_{\sqcup\!\sqcup} \\ &= e^{x_0 \log z} \mu[\mathrm{L}_{\mathrm{reg}}(1-z)] \mathrm{Mono}(z) Z_{\sqcup\!\sqcup}. \end{aligned}$$

Thus, $\mathrm{P}(z) \underset{z\to 0}{\widetilde{\quad}} e^{x_0 \log z}$ and $\mathrm{P}(z) \underset{z\to 1}{\widetilde{\quad}} \mathrm{Mono}(z) Z_{\sqcup\!\sqcup}$ yielding the expected results. □

As consequence of (48)-(50) and of Proposition 4, one gets

Theorem 4 (*à la Abel*, [39])**.**

$$\lim_{z\to1}\exp\left(y_1\log\frac{1}{1-z}\right)\Lambda(z)=\lim_{N\to\infty}\exp\left(\sum_{k\geq1}\mathrm{H}_{y_k}(N)\frac{(-y_1)^k}{k}\right)\mathrm{H}(N)=\pi_Y Z_{\sqcup\!\sqcup}.$$

Therefore, the knowledge of the ordinary Taylor expansion at 0 of the polylogarithmic functions $\{\mathrm{P}_w(1-z)\}_{w\in X^*}$ gives

Theorem 5 ([12])**.** *For all $g\in\mathcal{C}\{\mathrm{P}_w\}_{w\in Y^*}$, there exists algorithmically computable $c_j\in\mathbb{C},\alpha_j\in\mathbb{Z},\beta_j\in\mathbb{N}$ and $b_i\in\mathbb{C},\eta_i\in\mathbb{Z},\kappa_i\in\mathbb{N}$ such that*

$$g(z)\underset{z\to1}{\widetilde{\quad}}\sum_{j=0}^{+\infty}c_j(1-z)^{\alpha_j}\log^{\beta_j}(1-z)\quad\text{and}\quad[z^n]g(z)\underset{N\to+\infty}{\widetilde{\quad}}\sum_{i=0}^{+\infty}b_i n^{\eta_i}\log^{\kappa_i}(n).$$

Definition 3. *Let $\mathcal{Z}$ be the $\mathbb{Q}$-algebra generated by convergent polyzêtas and let $\mathcal{Z}'$ be the*[11] *$\mathbb{Q}[\gamma]$-algebra generated by $\mathcal{Z}$.*

Corollary 1 ([12])**.** *There exists algorithmically computable $c_j\in\mathcal{Z},\alpha_j\in\mathbb{Z},\beta_j\in\mathbb{N}$ and $b_i\in\mathcal{Z}',\kappa_i\in\mathbb{N},\eta_i\in\mathbb{Z}$ such that*

$$\begin{array}{lcll}\forall w\in Y^*,\quad \mathrm{P}_w(z) & \sim & \displaystyle\sum_{j=0}^{+\infty}c_j(1-z)^{\alpha_j}\log^{\beta_j}(1-z) & \text{for}\quad z\to1,\\ \forall w\in Y^*,\quad \mathrm{H}_w(N) & \sim & \displaystyle\sum_{i=0}^{+\infty}b_iN^{\eta_i}\log^{\kappa_i}(N) & \text{for}\quad N\to+\infty.\end{array}$$

The Chen generating series along the path $z_0\rightsquigarrow z$, associated to ω_0,ω_1 is the following

$$S_{z_0\rightsquigarrow z}:=\sum_{w\in X^*}\langle S\mid w\rangle\, w\quad\text{with}\quad\langle S\mid w\rangle=\alpha_{z_0}^z(w)\tag{51}$$

which solves the differential equation (34) with the initial condition $S_{z_0\rightsquigarrow z_0}=1$. Thus, $S_{z_0\rightsquigarrow z}$ and $\mathrm{L}(z)\mathrm{L}(z_0)^{-1}$ satisfy the same differential equation taking the same value at z_0 and

$$S_{z_0\rightsquigarrow z}=\mathrm{L}(z)\mathrm{L}(z_0)^{-1}.\tag{52}$$

Any Chen generating series $S_{z_0\rightsquigarrow z}$ is group like [44] and depends only on the homotopy class of $z_0\rightsquigarrow z$ [10]. The product of $S_{z_1\rightsquigarrow z_2}$ and $S_{z_0\rightsquigarrow z_1}$ is the Chen generating series

$$S_{z_0\rightsquigarrow z_2}=S_{z_1\rightsquigarrow z_2}S_{z_0\rightsquigarrow z_1}.\tag{53}$$

[11] Here, γ stands for the Euler constant $\gamma=.5772156649015328606065120900824024 3\ldots$

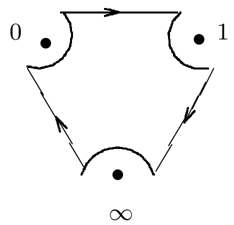

Figure 1: Hexagonal path

Let $\varepsilon \in]0,1[$ and let $z_i = \varepsilon \exp(\mathrm{i}\theta_i)$, for $i = 0$ or 1. We set $\theta = \theta_1 - \theta_0$. Let $\Gamma_0(\varepsilon,\theta)$ (resp. $\Gamma_1(\varepsilon,\theta)$) be the path turning around 0 (resp. 1) in the positive direction from z_0 to z_1. By induction on the length of w, one has

$$|\langle S_{\Gamma_i(\varepsilon,\theta)} \mid w\rangle| \quad = \quad (2\varepsilon)^{|w|_{x_i}} \theta^{|w|}/|w|!, \tag{54}$$

where, $|w|$ denotes the lenghth of w and $|w|_{x_i}$, denotes the number of occurrences of letter x_i in w, for $i = 0, 1$. For $\varepsilon \to 0^+$, these estimations yield

$$S_{\Gamma_i(\varepsilon,\theta)} \quad = \quad e^{\mathrm{i}\theta x_i} + o(\varepsilon). \tag{55}$$

In particular, if $\Gamma_0(\varepsilon)$ (resp. $\Gamma_1(\varepsilon)$) is a circular path of radius ε turning around 0 (resp. 1) in the positive direction, starting at $z = \varepsilon$ (resp. $1 - \varepsilon$), then, by the noncommutative residu theorem [35, 32], we get

$$S_{\Gamma_0(\varepsilon)} = e^{2\mathrm{i}\pi x_0} + o(\varepsilon) \quad \text{and} \quad S_{\Gamma_1(\varepsilon)} = e^{-2\mathrm{i}\pi x_1} + o(\varepsilon). \tag{56}$$

Finally, the asymptotic behaviors of L on (47) give

Proposition 5 ([32, 35])**.** *We have* $S_{\varepsilon\rightsquigarrow 1-\varepsilon} \underset{\varepsilon\to 0^+}{\widetilde{\quad}} e^{-x_1 \log\varepsilon} Z_{\sqcup\sqcup}\, e^{-x_0\log\varepsilon}$.

In other terms, $Z_{\sqcup\sqcup}$ is the regularized Chen generating series $S_{\varepsilon\rightsquigarrow 1-\varepsilon}$ of differential forms ω_0 and ω_1 : $Z_{\sqcup\sqcup}$ is the noncommutative generating series of the finite parts of the coefficients of the Chen generating series $e^{x_1\log\varepsilon}\ S_{\varepsilon\rightsquigarrow 1-\varepsilon}\ e^{x_0\log\varepsilon}$: the concatenation of $e^{x_0\log\varepsilon}$ and then $S_{\varepsilon\rightsquigarrow 1-\varepsilon}$ and finally, $e^{x_1\log\varepsilon}$.

Proposition 6. *Let $\rho_{1-1/z}$ be the morphism is given in Section 2.2.2. We have*

$$\prod_{\substack{l\in\mathcal{L}ynX\\ l\neq x_0,x_1}}^{\searrow} e^{\zeta(\check{l})l} \quad = \quad e^{\mathrm{i}\pi x_0} \prod_{\substack{l\in\mathcal{L}ynX\\ l\neq x_0,x_1}}^{\searrow} e^{\zeta(\check{l})\rho_{1-1/z}(l)} e^{\mathrm{i}\pi(-x_0+x_1)} \prod_{\substack{l\in\mathcal{L}ynX\\ l\neq x_0,x_1}}^{\searrow} e^{\zeta(\check{l})\rho^2_{1-1/z}(l)} e^{-\mathrm{i}\pi x_1}.$$

Proof. Following the hexagonal path given in Figure 1, one has [36, 32]

$$(S_{\varepsilon\rightsquigarrow 1-\varepsilon}e^{\mathrm{i}\pi x_0})\rho_{1-1/z}(S_{\varepsilon\rightsquigarrow 1-\varepsilon}e^{\mathrm{i}\pi x_0})\rho^2_{1-1/z}(S_{\varepsilon\rightsquigarrow 1-\varepsilon}e^{\mathrm{i}\pi x_0}) \quad = \quad 1 + O(\sqrt{\varepsilon}).$$

By Proposition 5, it follows the hexagonal relation [15, 16, 36, 32] which is

$$\begin{aligned} & Z_{\sqcup\sqcup} e^{\mathrm{i}\pi x_0}\rho_{1-1/z}(Z_{\sqcup\sqcup})e^{\mathrm{i}\pi(-x_0+x_1)}\rho^2_{1-1/z}(Z_{\sqcup\sqcup})e^{-\mathrm{i}\pi x_1} = 1,\\ \iff\quad & e^{\mathrm{i}\pi x_0}\rho_{1-1/z}(Z_{\sqcup\sqcup})e^{\mathrm{i}\pi(-x_0+x_1)}\rho^2_{1-1/z}(Z_{\sqcup\sqcup})e^{-\mathrm{i}\pi x_1} = Z^{-1}_{\sqcup\sqcup}. \end{aligned}$$

It follows then the expected result. ∎

2.3 Indiscernability over a class of formal power series

2.3.1 Residual calculus and representative series

Definition 4. *Let $S \in \mathbb{Q}\langle\langle X\rangle\rangle$ and let $P \in \mathbb{Q}\langle X\rangle$.*

The left residual *(resp.* right residual*) of S by P, is the formal power series $P \triangleleft S$ (resp. $S \triangleright P$) in $\mathbb{Q}\langle\langle X\rangle\rangle$ defined by :*

$$\langle P \triangleleft S \mid w\rangle = \langle S \mid wP\rangle \quad (resp. \quad \langle S \triangleright P \mid w\rangle = \langle S \mid Pw\rangle).$$

We straightforwardly get, for any $P, Q \in \mathbb{Q}\langle X\rangle$:

$$P \triangleleft (Q \triangleleft S) = PQ \triangleleft S,\ (S \triangleright P) \triangleright Q = S \triangleright PQ,\ (P \triangleleft S) \triangleright Q = P \triangleleft (S \triangleright Q). \quad (57)$$

In case $x, y \in X$ and $w \in X^*$, we get $x \triangleleft (wy) = \delta_{x,y} w$ and $xw \triangleright y = \delta_{x,y} w$.

Lemma 1 (Reconstruction lemma)**.** *Let $S \in \mathbb{Q}\langle\langle X\rangle\rangle$. Then*

$$S = \langle S \mid \epsilon\rangle + \sum_{x \in X} x(S \triangleright x) = \langle S \mid \epsilon\rangle + \sum_{x \in X} (x \triangleleft S)x.$$

Lemma 2. *The left and right residuals by a letter x are derivations in $(\mathbb{Q}\langle\langle X\rangle\rangle, ⧢)$:*

$$x \triangleleft (u ⧢ v) = (x \triangleleft u) ⧢ v + u ⧢ (x \triangleleft v), \qquad (u ⧢ v) \triangleright x = (u \triangleright x) ⧢ v + u ⧢ (v \triangleright x).$$

Proof. Use the recursive definitions of the shuffle product. □

Lemma 3. *For any Lie polynomial $Q \in \mathcal{L}ie_{\mathbb{Q}}\langle X\rangle$, the linear maps "$Q\triangleleft$" and "$\triangleright Q$" are derivations on $(\mathbb{Q}[\mathcal{L}ynX], ⧢)$.*

Proof. For any $l, l_1, l_2 \in \mathcal{L}ynX$, we have

$$\hat{l} \triangleleft (l_1 ⧢ l_2) = l_1 ⧢ (\hat{l} \triangleleft l_2) + (\hat{l} \triangleleft l_1) ⧢ l_2 = l_1 \delta_{l_2,\hat{l}} + \delta_{l_1,\hat{l}} l_2,$$
$$(l_1 ⧢ l_2) \triangleright \hat{l} = l_1 ⧢ (l_2 \triangleright \hat{l}) + (l_1 \triangleright \hat{l}) ⧢ l_2 = l_1 \delta_{l_2,\hat{l}} + \delta_{l_1,\hat{l}} l_2.$$

□

Lemma 4. *For any Lyndon word $l \in \mathcal{L}ynX$ and $\check{S}_l$ defined as in (39), one has*

$$x_1 \triangleleft l = l \triangleright x_0 = 0 \quad and \quad x_1 \triangleleft \check{S}_l = \check{S}_l \triangleright x_0 = 0.$$

Proof. Since $x_1 \triangleleft$ and $\triangleright x_0$ are derivations and for any $l \in \mathcal{L}ynX - X$, the polynomial $\check{S}_l$ belongs to $x_0 \mathbb{Q}\langle X\rangle x_1$ then it follows the expected results. □

Theorem 6 (On representative series)**.** *The following properties are equivalent for any series $S \in \mathbb{Q}\langle\langle X\rangle\rangle$:*

1. *The left $\mathbb{C}$-module $Res_g(S) = span\{w \triangleleft S \mid w \in X^*\}$ is finite dimensional.*
2. *The right $\mathbb{C}$-module $Res_d(S) = span\{S \triangleright w \mid w \in X^*\}$ is finite dimensional.*

3. *There are matrices $\lambda \in \mathcal{M}_{1,n}(\mathbb{Q})$, $\eta \in \mathcal{M}_{n,1}(\mathbb{Q})$ and a representation of X^* in $\mathcal{M}_{n,n}$, such that*

$$S = \sum_{w \in X^*} [\lambda\mu(w)\eta]\, w = \lambda \Big(\prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\mu(S_l)\, \check{S}_l} \Big) \eta.$$

A series that safisfies the items of this theorem will be called *representative series*. This concept can be found in [1, 42, 13]. The two first items are in [18, 24]. The third item can be deduced from [9, 11] for example and it was used to factorize first time, by Lyndon words, the output of bilinear and analytical dynamical systems respectively in [26, 27] and to study polylogarithms, hypergeometric functions and associated functions in [29, 31, 38]. The dimension of $Res_g(S)$ is equal to that of $Res_d(S)$, and to the minimal dimension of a representation satisfying the third point of Theorem 6. This rank is then equal to the rank of the Hankel matrix of S, that is the infinite matrix $(\langle S \mid uv \rangle)_{u,v \in X}$ indexed by $X^* \times X^*$ and is also called *Hankel rank* of S [18, 24] :

Definition 5 ([18, 24])**.** *The* Hankel rank *of a formal power series $S \in \mathbb{C}\langle\langle X \rangle\rangle$ is the dimension of the vector space*

$$\{S \triangleright \Pi \mid \Pi \in \mathbb{C}\langle X \rangle\}, \quad (\textit{resp.} \quad \{\Pi \triangleleft S \mid \Pi \in \mathbb{C}\langle X \rangle\}.$$

The triplet (λ, μ, η) is called a *linear representation* of S. We define the minimal representation[12] of S as being a representation of S of minimal dimension.

For any proper series S, the following power series is called "star of S"

$$S^* \quad = \quad 1 + S + S^2 + \ldots + S^n + \ldots . \tag{58}$$

Definition 6 ([2, 47])**.** *A series S is called* rational *if it belongs to the closure in $\mathbb{Q}\langle\langle X \rangle\rangle$ of the noncommutative polynomial algebra by sum, product and star operation of* proper[13] *elements. The set of rational power series will be denoted by $\mathbb{Q}^{\mathrm{rat}}\langle\langle X \rangle\rangle$.*

Lemma 5. *For any noncommutative rational series (resp. polynomial) R and for any polynomial P, the left and right residuals of R by P are rational (resp. polynomial).*

Theorem 7 (Schützenberger, [2, 47])**.** *Any noncommutative power series is representative if and only if it is rational.*

2.3.2 Continuity and indiscernability

Definition 7 ([25, 39])**.** *Let $\mathcal{H}$ be a class of formal power series over X and let $S \in \mathbb{C}\langle\langle X \rangle\rangle$.*

[12] It can be shown that all minimal representations are isomorphics (see [2]).

[13] A series S is said to be proper if $\langle S \mid \epsilon \rangle = 0$.

1. *S is said to be* continuous[14] *over $\mathcal{H}$ if for any $\Phi \in \mathcal{H}$, the following sum, denoted $\langle S \parallel \Phi \rangle$, is convergent in norm*

$$\sum_{w \in X^*} \langle S \mid w \rangle \langle \Phi \mid w \rangle.$$

The set of continuous power series over $\mathcal{H}$ will be denoted by $\mathbb{C}^{\mathrm{cont}}\langle\langle X \rangle\rangle$.

2. *S is said to be* indiscernable[15] *over $\mathcal{H}$ if and only if*

$$\forall \Phi \in \mathcal{H}, \qquad \langle S \parallel \Phi \rangle = 0.$$

Let ρ be the monoid morphism verifying $\rho(x_0) = x_1$ and $\rho(x_1) = x_0$ and let $\hat{w} = \rho(\tilde{w})$, where $\tilde{w}$ is the mirror of w.

Lemma 6. *Let $S \in \mathbb{C}^{\mathrm{cont}}\langle\langle X \rangle\rangle$. If $\langle S \parallel Z_{\sqcup\!\sqcup} \rangle = 0$ then $\langle \hat{S} \parallel Z_{\sqcup\!\sqcup} \rangle = 0$, where*

$$\hat{S} \quad := \quad \sum_{w \in X^*} \langle S \mid w \rangle \; \hat{w}.$$

Proof. For any $w \in x_0 X^* x_1$, by "duality relation", one has (see [40, 50, 36])

$$\zeta(\hat{w}) = \zeta(w), \quad \text{or equivalently} \quad Z_{\sqcup\!\sqcup} = \hat{Z}_{\sqcup\!\sqcup} := \sum_{w \in X^*} \langle Z_{\sqcup\!\sqcup} \mid w \rangle \; \hat{w}.$$

Using the fact

$$\langle \hat{S} \parallel Z_{\sqcup\!\sqcup} \rangle = \sum_{\hat{w} \in X^*} \langle S \mid \hat{w} \rangle \langle Z_{\sqcup\!\sqcup} \mid \hat{w} \rangle = \sum_{w \in X^*} \langle S \mid w \rangle \langle Z_{\sqcup\!\sqcup} \mid w \rangle,$$

one gets finally the expected result. $\square$

Lemma 7. *Let $\mathcal{H}$ be a monoid containing $\{e^{t\,x}\}_{x \in X}^{t \in \mathbb{C}}$. Let $S \in \mathbb{C}^{\mathrm{cont}}\langle\langle X \rangle\rangle$ being indiscernable over $\mathcal{H}$. Then for any $x \in X$, $x \triangleleft S$ and $S \triangleright x$ belong to $\mathbb{C}^{\mathrm{cont}}\langle\langle X \rangle\rangle$ and they are indiscernable over $\mathcal{H}$.*

Proof. Let us calculate $\langle x \triangleleft S \parallel \Phi \rangle = \langle S \parallel \Phi x \rangle$ and $\langle S \triangleright x \parallel \Phi \rangle = \langle S \parallel x\Phi \rangle$. Since

$$\lim_{t \to 0} \frac{e^{t\,x} - 1}{t} = x \quad \text{and} \quad \lim_{t \to 0} \frac{e^{t\,x} - 1}{t} = x$$

then, for any $\Phi \in \mathcal{H}$, by uniform convergence, one has

$$\langle S \parallel \Phi x \rangle = \langle S \parallel \lim_{t \to 0} \Phi \frac{e^{t\,x} - 1}{t} \rangle = \lim_{t \to 0} \langle S \parallel \Phi \frac{e^{t\,x} - 1}{t} \rangle,$$
$$\langle S \parallel x\Phi \rangle = \langle S \parallel \lim_{t \to 0} \frac{e^{t\,x} - 1}{t} \Phi \rangle = \lim_{t \to 0} \langle S \parallel \frac{e^{t\,x} - 1}{t} \Phi \rangle.$$

[14] See [25, 39], for a convergence criterion and an example of continuous generating series.
[15] Here, we adapt this notion developped in [25] via the residual calculus.

Since S is indiscernable over $\mathcal{H}$ then

$$\langle S \parallel \Phi x\rangle = \lim_{t\to 0}\frac{1}{t}\langle S \parallel \Phi e^{t\,x}\rangle - \lim_{t\to 0}\frac{1}{t}\langle S \parallel \Phi\rangle = 0,$$
$$\langle S \parallel x\Phi\rangle = \lim_{t\to 0}\frac{1}{t}\langle S \parallel e^{t\,x}\Phi\rangle - \lim_{t\to 0}\frac{1}{t}\langle S \parallel \Phi\rangle = 0.$$

□

Proposition 7. *Let $\mathcal{H}$ be a monoid containing $\{e^{t\,x}\}_{x\in X}^{t\in\mathbb{C}}$. The formal power series $S \in \mathbb{C}^{\mathrm{cont}}\langle\langle X\rangle\rangle$ is indiscernable over $\mathcal{H}$ if and only if $S = 0$.*

Proof. If $S = 0$ then it is immediate that S is indiscernable over $\mathcal{H}$. Conversely, if S is indiscernable over $\mathcal{H}$ then by Lemma 7, for any word $w \in X^*$, by induction on the length of w, $w \triangleleft S$ is indiscernable over $\mathcal{H}$ and then in particular,

$$\langle w \triangleleft S \parallel \mathrm{Id}_{\mathcal{H}}\rangle = \langle S \mid w\rangle = 0.$$

In other words, $S = 0$. □

3 Group of associators : polynomial relations among convergent polyzêtas and identification of local coordinates

3.1 Generalized Euler constants and global regularization of polyzêtas

3.1.1 Three regularizations of divergent polyzêtas

Theorem 8 ([33]). *Let $\zeta_{⧢} : (\mathbb{Q}\langle Y\rangle, ⧢) \to (\mathbb{R}, .)$ be the morphism verifying the following properties*

- *for $u, v \in Y^*, \zeta_{⧢}(u ⧢ v) = \zeta_{⧢}(u)\zeta_{⧢}(v)$,*
- *for all convergent word $w \in Y^* - y_1Y^*, \zeta_{⧢}(w) = \zeta(w)$,*
- *$\zeta_{⧢}(y_1) = 0$.*

Then

$$\sum_{w\in X^*} \zeta_{⧢}(w)\, w \quad = \quad Z_{⧢}.$$

Corollary 2 ([33]). *For any $w \in X^*, \zeta_{⧢}(w)$ belongs to the algebra $\mathcal{Z}$.*

Theorem 9 ([33]). *Let $\zeta_{\sqcup\!\sqcup} : (\mathbb{Q}\langle X\rangle, \sqcup\!\sqcup) \to (\mathbb{R}, .)$ be the morphism verifying the following properties*

- *for $u, v \in X^*, \zeta_{\sqcup\!\sqcup}(u \sqcup\!\sqcup v) = \zeta_{\sqcup\!\sqcup}(u)\zeta_{\sqcup\!\sqcup}(v)$,*

- *for all convergent word $w \in x_0X^*x_1, \zeta_{\sqcup\sqcup}(w) = \zeta(w)$,*
- $\zeta_{\sqcup\sqcup}(x_0) = \zeta_{\sqcup\sqcup}(x_1) = 0.$

Then

$$\sum_{w\in X^*} \zeta_{\sqcup\sqcup}(w)\, w \quad = \quad Z_{\sqcup\sqcup}.$$

Corollary 3 ([33]). *For any $w \in Y^*, \zeta_{\sqcup\sqcup}(w)$ belongs to the algebra $\mathcal{Z}$.*

Definition 8. *For any $w \in Y^*$, let γ_w be the constant part*[16] *of the asymptotic expasion, on the comparison scale $\{n^a \log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$, of $\mathrm{H}_w(n)$.*

Let Z_γ be the noncommutative generating series of $\{\gamma_w\}_{w\in Y^*}$:

$$Z_\gamma \quad := \quad \sum_{w\in Y^*} \gamma_w\, w.$$

Definition 9. *We set*

$$B(y_1) := \exp\biggl(-\sum_{k\geq 1}\gamma_{y_k}\frac{(-y_1)^k}{k}\biggr) \quad \textit{and} \quad B'(y_1) := e^{-\gamma y_1}B(y_1).$$

The power series $B'(y_1)$ corresponds in fact to the mould[17] Mono in [17] and to the Φ_{corr} in [43] (see also [4, 8]). While the power series $B(y_1)$ corresponds to the Gamma Euler function with its product expansion,

$$B(y_1) = \Gamma(y_1+1), \qquad \frac{1}{\Gamma(y_1+1)} = e^{\gamma y_1}\prod_{n\geq 1}\Bigl(1+\frac{y_1}{n}\Bigr)e^{-\gamma/n}. \tag{59}$$

Lemma 8 ([39]). *Let $b_{n,k}(t_1,\ldots,t_{n-k+1})$ be the (exponential) partial Bell polynomials in the variables $\{t_l\}_{l\geq 1}$ given by the exponential generating series*

$$\exp\biggl(u\sum_{l=0}^{\infty}t_l\frac{v^l}{l!}\biggr) \quad = \quad \sum_{n,k=0}^{\infty} b_{n,k}(t_1,\ldots,t_{n-k+1})\frac{v^n u^k}{n!}.$$

For any $m \geq 1$, let $t_m = (-1)^m(m-1)!\gamma_{y_m}$. Then

$$B(y_1) \quad = \quad 1 + \sum_{n\geq 1}\biggl(\sum_{k=1}^{n} b_{n,k}(\gamma, -\zeta(2), 2\zeta(3), \ldots)\biggr)\frac{(-y_1)^n}{n!}.$$

Since the ordinary generating series of the finite parts of coefficients of $\mathrm{Const}(N)$ is nothing else but the power series $B(y_1)$, taking the constant part on either side of $\mathrm{H}(N) \underset{N\to\infty}{\widetilde{\quad}} \mathrm{Const}(N)\pi_Y Z_{\sqcup\sqcup}$ (see Proposition 4), yields

Theorem 10 ([39]). *We have $Z_\gamma = B(y_1)\pi_Y Z_{\sqcup\sqcup}$.*

[16] *i.e.* γ_w is the Euler-Mac Laurin constante of $\mathrm{H}_w(n)$.

[17] The readers can see why we have introduced the power series $\mathrm{Mono}(z)$ in Proposition 4.

Identifying the coefficients of $y_1^k w$ on either side using the identity[18] [33]

$$\forall u \in X^* x_1, \quad x_1^k x_0 u \quad = \quad \sum_{l=0}^{k} x_1^l \sqcup\!\sqcup (x_0[(-x_1)^{k-l} \sqcup\!\sqcup u]) \tag{60}$$

and applying the morphism $\zeta_{\sqcup\!\sqcup}$ given in Theorem 9, we get [33]

$$\forall u \in X^* x_1, \quad \zeta_{\sqcup\!\sqcup}(x_1^k x_0 u) \quad = \quad \zeta(x_0[(-x_1)^k \sqcup\!\sqcup u]). \tag{61}$$

Corollary 4 ([39]). *For $w \in x_0 X^* x_1$,* i.e. *$w = x_0 u$ and $\pi_Y w \in Y^* - y_1 Y^*$, and for $k \geq 0$, the constant $\gamma_{⧢}(x_1^k w)$ associated to the divergent polyzêta $\zeta(x_1^k w)$ is a polynomial of degree k in γ and with coefficients in $\mathcal{Z}$:*

$$\gamma_{x_1^k w} \quad = \quad \sum_{i=0}^{k} \frac{\zeta(x_0[(-x_1)^{k-i} \sqcup\!\sqcup u])}{i!} \left(\sum_{j=1}^{i} b_{i,j}(\gamma, -\zeta(2), 2\zeta(3), \ldots) \right).$$

Moreover, for $l = 0, .., k$, the coefficient of γ^l is of weight $\mid w \mid +k - l$.

In particular, for $s > 1$, the constant $\gamma_{y_1 y_s}$ associated to $\zeta(y_1 y_s)$ is linear in γ and with coefficients in $\mathbb{Q}[\zeta(2), \zeta(2i+1)]_{0<i\leq(s-1)/2}$.

Corollary 5 ([39]). *The constant $\gamma_{x_1^k}$ associated to the divergent polyzêta $\zeta(x_1^k)$ is a polynomial of degree k in γ with coefficients in $\mathbb{Q}[\zeta(2), \zeta(2i+1)]_{0<i\leq(k-1)/2}$:*

$$\gamma_{x_1^k} \quad = \quad \sum_{\substack{s_1,\ldots,s_k \geq 0 \\ s_1+\ldots+ks_k=k+1}} \frac{(-1)^k}{s_1! \ldots s_k!} (-\gamma)^{s_1} \left(-\frac{\zeta(2)}{2} \right)^{s_2} \ldots \left(-\frac{\zeta(k)}{k} \right)^{s_k}.$$

Moreover, for $l = 0, .., k$, the coefficient of γ^l is of weight $k - l$.

We thereby obtain the following algebra morphism, denoted by $\gamma_\bullet$, for the regularization to γ with respect to the quasi-shuffle product *independently* to the regularization with respect to the shuffle product[19] and then by applying the tensor product of morphisms $\gamma_\bullet \otimes \mathrm{Id}$ on the diagonal series, over Y, we get (see Annexe A)

Theorem 11. *The mapping $\gamma_\bullet$ realizes the morphism from $(\mathbb{Q}\langle Y \rangle, ⧢)$ to $(\mathbb{R}, .)$ verifying the following properties*

[18] By the Convolution Theorem [28], this is equivalent to

$$\begin{aligned} \forall u \in X^*, \quad \alpha_0^z(x_1^k x_0 u) \quad &= \quad \int_0^z \frac{[\log(1-s) - \log(1-z)]^k}{k!} \alpha_0^s(u) \frac{ds}{s} \\ &= \quad \sum_{l=0}^{k} \frac{[-\log(1-z)]^l}{l!} \int_0^z \frac{\log^{k-l}(1-s)}{(k-l)!} \alpha_0^s(u) \frac{ds}{s}. \end{aligned}$$

This theorem induces *de facto* the algebra morphism of regularization to 0 with respect to the shuffle product, as shown the Theorem 9.

[19] In [4, 8, 21, 48], the authors suggest the *simultaneous* regularizations, with respect to the shuffle product and the quasi-shuffle product, to T and then to set $T = 0$.

- *for any word* $u, v \in Y^*, \gamma_{u \,\text{⧢}\, v} = \gamma_u \gamma_v$,
- *for any convergent word* $w \in Y^* - y_1 Y^*, \gamma_w = \zeta(w)$
- $\gamma_{y_1} = \gamma$.

Then $Z_\gamma = e^{\gamma y_1} Z_{\text{⧢}}$.

3.1.2 Identities of noncommutative generating series of polyzêtas

Corollary 6. *With the notations of Definition 9, we have*

$$\begin{aligned} Z_\gamma = B(y_1)\pi_Y Z_{\sqcup\!\sqcup} &\iff Z_{\text{⧢}} = B'(y_1)\pi_Y Z_{\sqcup\!\sqcup}, \\ \pi_Y Z_{\sqcup\!\sqcup} = B^{-1}(x_1) Z_\gamma &\iff Z_{\sqcup\!\sqcup} = B'^{-1}(x_1)\pi_X Z_{\text{⧢}}. \end{aligned}$$

Roughly speaking, for the quasi-shuffle product, the regularization to γ is "equivalent" to the regularization to 0.

Note also that the constant $\gamma_{y_1} = \gamma$ is obtained as the finite part of the asymptotic expansion of $\mathrm{H}_1(n)$ in the comparison scale $\{n^a \log^b(n)\}_{a\in\mathbb{Z}, b\in\mathbb{N}}$.

In the same way, since n and $\mathrm{H}_1(n)$ are algebraically independent, as arithmetical functions (see Proposition 1), then $\{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z}, b\in\mathbb{N}}$ constitutes a new comparison scale for asymptotic expansions.

Hence, the constants $\zeta_{\sqcup\!\sqcup}(x_1) = 0$ and $\zeta_{\text{⧢}}(y_1) = 0$ can be interpreted as the finite part of the asymptotic expansions of $\mathrm{Li}_1(z)$ and $\mathrm{H}_1(n)$ respectively in the comparison scales $\{(1-z)^a \log(1-z)^b\}_{a\in\mathbb{Z}, b\in\mathbb{N}}$ and $\{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z}, b\in\mathbb{N}}$.

Definition 10 ([33])**.** *Let* $C_1 := \mathbb{Q}\epsilon \oplus x_0 \mathbb{Q}\langle X\rangle x_1, C_2 := \mathbb{Q}\epsilon \oplus (Y - \{y_1\})\mathbb{Q}\langle Y\rangle$.

Lemma 9 ([32, 33])**.** *We get* $(C_1, \sqcup\!\sqcup) \cong (C_2, \text{⧢})$.

Using a theorem of Radford [45] and its analogous over Y (see Annexe A), we get

Proposition 8 ([32, 33])**.**

$$\begin{aligned} (\mathbb{Q}\langle X\rangle, \sqcup\!\sqcup) &\cong (\mathbb{Q}[\mathcal{L}ynX], \sqcup\!\sqcup) = C_1[x_0, x_1], \\ (\mathbb{Q}\langle Y\rangle, \text{⧢}) &\cong (\mathbb{Q}[\mathcal{L}ynY], \text{⧢}) = C_2[y_1], \end{aligned}$$

This insures the effective way to get the finite part of the asymptotic expansions, in the comparison scales $\{(1-z)^a \log(1-z)^b\}_{a\in\mathbb{Z}, b\in\mathbb{N}}$ and $\{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z}, b\in\mathbb{N}}$, of $\{\mathrm{Li}_w(z)\}_{w\in Y^*}$ and $\{\mathrm{H}_w(N)\}_{w\in Y^*}$ respectively.

Proposition 9 ([32, 33])**.** *The restrictions of* $\zeta_{\sqcup\!\sqcup}$ *and* $\zeta_{\text{⧢}}$ *over* $(C_1, \sqcup\!\sqcup)$ *and* $(C_2, \text{⧢})$ *respectively coincide with the following* surjective *algebra morphism*

$$\begin{array}{rcl} \zeta \quad : \quad \begin{array}{c}(C_2, \text{⧢})\\ (C_1, \sqcup\!\sqcup)\end{array} & \longrightarrow & (\mathbb{R}, .) \\ \begin{array}{c} y_{r_1}\ldots y_{r_k} \\ x_0 x_1^{r_1-1}\ldots x_0 x_1^{r_k-1}\end{array} & \longmapsto & \displaystyle\sum_{n_1>\ldots>n_k>0} \frac{1}{n_1^{r_1}\ldots n_k^{r_k}}, \end{array}$$

In Section 3.3 we will give the complete description of the kernel $\ker\zeta$.

With the double regularization[20] to zero [4, 8, 33, 43], the Drindfel'd associator Φ_{KZ} corresponds then to $Z_{⧢}$ (obtained with only convergent polyzêtas) as being the unique group-like element satisfying [35, 32]

$$\langle Z_{⧢} \mid x_0\rangle = \langle Z_{⧢} \mid x_1\rangle = 0 \quad \text{and} \quad \forall x \in x_0X^*x_1, \quad \langle Z_{⧢} \mid w\rangle = \zeta(w). \tag{62}$$

As consequence of Proposition 2, one has

Proposition 10 ([38]).

$$\begin{aligned} \log Z_{⧢} &= \sum_{w\in X^*} \zeta_{⧢}(w)\ \pi_1(w), \\ &= \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1,\ldots,u_k\in X^*-\{\epsilon\}} \zeta_{⧢}(u_1 ⧢ \ldots ⧢ u_k)\ u_1\ldots u_k. \end{aligned}$$

The associator Φ_{KZ} can be also graded in the adjoint basis of $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X\rangle)$ as follows

Proposition 11 ([38]). *For any $l \in \mathbb{N}$ and $P \in \mathbb{C}\langle X\rangle$, let $\circ$ denotes the composite operation defined by $x_1x_0^l \circ P = x_1(x_0^l ⧢ P)$. Then*

$$Z_{⧢} = \sum_{k\geq 0} \sum_{l_1,\cdots,l_k\geq 0} \zeta_{⧢}(x_1x_0^{l_1} \circ \cdots \circ x_1x_0^{l_k}) \prod_{i=0}^{k} \mathrm{ad}_{x_0}^{l_i} x_1,$$

where $\mathrm{ad}_{x_0}^{l} x_1$ *is iterated Lie bracket* $\mathrm{ad}_{x_0}^{l} x_1 = [x_0, \mathrm{ad}_{x_0}^{l-1} x_1]$ *and* $\mathrm{ad}_{x_0}^{0} x_1 = x_1$.

Using the following expansion [6]

$$\mathrm{ad}_{x_0}^{n} x_1 = \sum_{i=0}^{n} \binom{i}{n} x_0^{n-i}x_1x_0^{i}, \tag{63}$$

one deduces then, via the regularization process of Theorem 9, the expression of the Drindfel'd associator Φ_{KZ} given by Lê and Murakami [22].

3.2 Action of differential Galois group of polylogarithms on their asymptotic expansions

3.2.1 Group of associators theorem

Let A a be a commutative $\mathbb{Q}$-algebra.

Since the polyzêtas satisfy (36), then by the Friedrichs criterion we can state the following

[20] This double regularization is deduced from of the noncommutative generating series $Z_{⧢}$ and $Z_{⧢\!\!\!-}$ on the definitions 1 and 2 (see the theorems 8 and 9).

Definition 11. *Let $dm(A)$ be the set of $\Phi \in A\langle\langle X\rangle\rangle$ such that*[21] $\langle \Phi \mid \epsilon\rangle = 1$, $\langle \Phi \mid x_0\rangle = \langle \Phi \mid x_1\rangle = 0, \Delta_{\sqcup\!\sqcup}\Phi = \Phi \otimes \Phi$ *and such that, for*

$$\Psi = B'(y_1)\pi_Y \Phi \quad \in \quad A\langle\langle Y\rangle\rangle$$

then[22] $\Delta_{\uplus}\Psi = \Psi \otimes \Psi$.

Proposition 12 ([38])**.** *If $G(z)$ and $H(z)$ are exponential solutions of (DE) then there exsists a Lie series $C \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle$ such that $G(z) = H(z)\exp(C)$.*

Proof. Since $H(z)H(z)^{-1} = 1$ then by differentiating, we have

$$d[H(z)]H(z)^{-1} \quad = \quad -H(z)d[H(z)^{-1}].$$

Therefore if $H(z)$ is solution of Drinfel'd equation then

$$\begin{aligned} d[H(z)^{-1}] &= -H(z)^{-1}[dH(z)]H(z)^{-1} \\ &= -H(z)^{-1}[x_0\omega_0(z) + x_1\omega_1(z)], \\ d[H(z)^{-1}G(z)] &= H(z)^{-1}[dG(z)] + [dH(z)^{-1}]G(z) \\ &= H(z)^{-1}[x_0\omega_0(z) + x_1\omega_1(z)]G(z) \\ &- H(z)^{-1}[x_0\omega_0(z) + x_1\omega_1(z)]G(z). \end{aligned}$$

By simplification, we deduce then $H(z)^{-1}G(z)$ is a constant formal power series. Since the inverse and the product of group like elements is group like then we get the expected result. □

The differential $\mathcal{C}$-module $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ is the universal Picard-Vessiot extension of every linear differential equations, with coefficients in $\mathcal{C}$ and admitting $\{0, 1, \infty\}$ as regular singularities. The universal differential Galois group, noted by $\mathrm{Gal}(\mathrm{LI}_{\mathcal{C}})$, is the set of differential $\mathcal{C}$-automorphisms of $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ (*i.e* the automorphisms of $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ that let $\mathcal{C}$ point-wise fixed and that commute with derivation). The action of an automorphism of $\mathrm{Gal}(\mathrm{LI}_{\mathcal{C}})$ can be determined by its action on Li_w, for $w \in X^*$. It can be resumed as its action on the noncommutative generating series L [38] :

Let $\sigma \in \mathrm{Gal}(\mathrm{LI}_{\mathcal{C}})$. Then

$$\sum_{w\in X^*} \sigma\,\mathrm{Li}_w\ w \quad = \quad \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\sigma\,\mathrm{Li}_{\check{S}_l}\, S_l}. \tag{64}$$

Since $d\sigma\,\mathrm{Li}_{x_i} = \sigma d\,\mathrm{Li}_{x_i} = \omega_i$ then by integrating the two memmbers, we obtain $\sigma\,\mathrm{Li}_{x_i} = \mathrm{Li}_{x_i} + c_{x_i}$, where c_{x_i} is a constant of integration. More generally, for any Lyndon word $l = x_i l_1^{i_1} \cdots l_k^{i_k}$ with $l_1 > \cdots > l_k$, one has

$$\sigma\,\mathrm{Li}_{\check{S}_l} \quad = \quad \int \omega_{x_i} \frac{\sigma\,\mathrm{Li}_{\check{S}_{l_1}}^{i_1}}{i_1!} \cdots \frac{\sigma\,\mathrm{Li}_{\check{S}_{l_k}}^{i_k}}{i_k!} + c_{\check{S}_l}, \tag{65}$$

[21] $\Delta_{\sqcup\!\sqcup}$ denotes the co-product of the shuflle product.

[22] $\Delta_{\uplus}$ denotes the co-product of the quasi-shuflle product.

where $c_{\check{S}_l}$ is a constant of integration. For example,

$$\sigma\,\mathrm{Li}_{x_0x_1} = \mathrm{Li}_{x_0x_1} + c_{x_1}\mathrm{Li}_{x_0} + c_{x_0x_1}, \tag{66}$$

$$\sigma\,\mathrm{Li}_{x_0^2x_1} = \mathrm{Li}_{x_0^2x_1} + \frac{c_{x_1}}{2}\mathrm{Li}_{x_0}^2 + c_{x_0x_1}\mathrm{Li}_{x_0} + c_{x_0^2x_1}, \tag{67}$$

$$\sigma\,\mathrm{Li}_{x_0x_1^2} = \mathrm{Li}_{x_0x_1^2} + c_{x_1}\mathrm{Li}_{x_0x_1} + \frac{c_{x_1}^2}{2}\mathrm{Li}_{x_0} + c_{x_0x_1^2}. \tag{68}$$

Consequently,

$$\sum_{w\in X^*}\sigma\,\mathrm{Li}_w\;w = \mathrm{L}e^{C_\sigma} \quad \text{where} \quad e^{C_\sigma} := \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{c_{\check{S}_l}S_l}. \tag{69}$$

The action of $\sigma\in\mathrm{Gal}(\mathrm{LI}_{\mathcal{C}})$ over $\{\mathrm{Li}_w\}_{w\in X^*}$ is then equivalent to the action of the Lie exponential $e^{C_\sigma}\in\mathrm{Gal}(DE)$ over the exponential solution L. So,

Theorem 12 ([38])**.** *We have* $\mathrm{Gal}(\mathrm{LI}_{\mathcal{C}}) = \{e^C \mid C\in\mathcal{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle\}$.

Typically, since $\mathrm{L}(z_0)^{-1}$ is group-like then $S_{z_0\rightsquigarrow z} = \mathrm{L}(z)\mathrm{L}(z_0)^{-1}$ is an other solution of (34) as already saw in (52).

Theorem 13 (Group of associators theorem)**.** *Let $\Phi\in A\langle\langle X\rangle\rangle$ and $\Psi\in A\langle\langle Y\rangle\rangle$ be group-like elements, for the co-products $\Delta_{\sqcup\!\sqcup}, \Delta_{\uplus}$ respectively, such that $\Psi = B(y_1)\pi_Y\Phi$. There exists an unique $C\in\mathcal{L}ie_A\langle\langle X\rangle\rangle$ such that $\Phi = Z_{\sqcup\!\sqcup}e^C$ and $\Psi = B(y_1)\pi_Y(Z_{\sqcup\!\sqcup}e^C)$.*

Proof. If $C\in\mathcal{L}ie_A\langle\langle X\rangle\rangle$ then $\mathrm{L}' = \mathrm{L}e^C$ is group-like, for the co-product $\Delta_{\sqcup\!\sqcup}$, and $e^C\in\mathrm{Gal}(DE)$. Let H′ be the noncommutative generating series of the Taylor coefficients, belonging to the harmonic algebra, of $\{(1-z)^{-1}\langle\mathrm{L}'\mid w\rangle\}_{w\in Y^*}$. Then $\mathrm{H}'(N)$ is also group-like, for the co-product $\Delta_{\uplus}$. By the asymptotic expansion of L, we have $\mathrm{L}'(z)\,\widetilde{_{\varepsilon\to 1}}\,e^{-x_1\log(1-z)}Z_{\sqcup\!\sqcup}e^C$ [36, 32]. We put then $\Phi := Z_{\sqcup\!\sqcup}e^C$ and we deduce that

$$\frac{\mathrm{L}'(z)}{1-z}\;\widetilde{_{z\to 1}}\;\mathrm{Mono}(z)\Phi \quad\text{and}\quad \mathrm{H}'(N)\;\widetilde{_{N\to\infty}}\;\mathrm{Const}(N)\pi_Y\Phi,$$

where the expressions of $\mathrm{Mono}(z)$ and $\mathrm{Const}(N)$ are given on (49) and (50) respectively. Let κ_w be the constant part of $\mathrm{H}'_w(N)$. Then

$$\sum_{w\in Y^*}\kappa_w\,w = B(y_1)\pi_Y\Phi.$$

We put then $\Psi := B(y_1)\pi_Y\Phi$ (and also $\Psi' := B'(y_1)\pi_Y\Phi$). ☐

Corollary 7. *We have*

$$dm(A) = \{Z_{\sqcup\!\sqcup}e^C \mid C\in\mathcal{L}ie_A\langle\langle X\rangle\rangle \quad \textit{and}\ \langle e^C\mid\epsilon\rangle = 1, \langle e^C\mid x_0\rangle = \langle e^C\mid x_1\rangle = 0\}.$$

Proof. On the one hand, $\langle\Phi\mid x_0\rangle = \langle Z_{\sqcup\!\sqcup}\mid x_0\rangle = 0$, $\langle\Phi\mid x_1\rangle = \langle Z_{\sqcup\!\sqcup}\mid x_1\rangle = 0$ and on the other, $\langle\Phi\mid\epsilon\rangle = \langle Z_{\sqcup\!\sqcup}\mid\epsilon\rangle = 1$, the result follows. ☐

Note also that if $\mathcal{Z} \subset A$ then $dm(A)$ forms a group and with the notations of Corollary 6, we obtain

Corollary 8. *For any associator $\Phi = Z_{\sqcup\!\sqcup} e^C \in dm(A)$, let $\Psi = B(y_1)\pi_Y\Phi$ and let $\Psi' = B'(y_1)\pi_Y\Phi$. Then*

$$\Psi = B(y_1)\pi_Y\Phi \iff \Psi' = B'(y_1)\pi_Y\Phi.$$

Proof. Since Ψ is group like and since $\langle\Phi \mid x_1\rangle = \langle\Psi' \mid y_1\rangle = 0$ and $\langle\Psi \mid y_1\rangle = \gamma$ then, using the factorization by Lyndon words, we get the expected result. □

Lemma 10. *Let $\Phi = Z_{\sqcup\!\sqcup} e^C \in dm(A)$ and let $\Psi = B(y_1)\pi_Y(Z_{\sqcup\!\sqcup} e^C)$. The local coordinates (of second kind) of Φ (resp. Ψ) are polynomials on $\{\zeta_{\sqcup\!\sqcup}(\check{S}_l)\}_{l\in\mathcal{L}ynX}$ (resp. $\{\zeta_{\text{⧢}}(\check{\Sigma}_l)\}_{l\in\mathcal{L}ynY}$) of $\mathcal{Z}$ (resp. $\mathcal{Z}'$). While C describes $\mathcal{L}ie_A\langle\langle X\rangle\rangle$, these coordinates describe $A[\{\zeta_{\sqcup\!\sqcup}(\check{S}_l)\}_{l\in\mathcal{L}ynX}]$ (resp. $A[\{\zeta_{\text{⧢}}(\check{\Sigma}_l)\}_{l\in\mathcal{L}ynY}]$).*

Proof. Let $\Phi \in dm(A)$. By Corollary 7, there exists $P \in \mathcal{L}ie_A\langle\langle X\rangle\rangle$ verifying $\langle e^P \mid \epsilon\rangle = 1, \langle e^P \mid x_0\rangle = \langle e^P \mid x_1\rangle = 0$ such that $\Phi = Z_{\sqcup\!\sqcup} e^P$. Using the factorization forms by Lyndon words, we get

$$\prod_{l\in\mathcal{L}ynX-X}^{\searrow} e^{\phi(\check{S}_l)\; S_l} = \Big(\prod_{l\in\mathcal{L}ynX-X}^{\searrow} e^{\zeta(\check{S}_l)\; S_l}\Big)\Big(\prod_{l\in\mathcal{L}ynX-X}^{\searrow} e^{p_{\check{S}_l}\; S_l}\Big).$$

Expanding the Hausdorff product and identifying the local coordinates in the PBW-Lyndon basis there exists $I_l \subset \{\lambda \in \mathcal{L}ynX - X \text{ s.t. } |\lambda| \le |l|\}$, for $l \in \mathcal{L}ynX - X$, and the coefficients $\{p'_{\check{S}_u}\}_{u\in I_l}$ belonging to A such that

$$\phi(\check{S}_l) = \sum_{u\in I_l} p'_{\check{S}_u}\, \zeta(\check{S}_u).$$

This belongs to $A[\{\zeta(\check{S}_l)\}_{l\in\mathcal{L}ynX-X}]$ and holds for any $P \in \mathcal{L}ie_A\langle\langle X\rangle\rangle$. □

With the notations of Definition 9 and by Corollary 8, we get in particular

Lemma 11. *For any $\Phi \in dm(A)$, by identifying the local coordinates (of second kind) on two members of the identities $\Psi = B(y_1)\pi_Y\Phi$, or equivalently of $\Psi' = B'(y_1)\pi_Y\Phi$, we get polynomial relations, of coefficients in A, among generators of the A-algebra of convergent polyzêtas.*

Therefore,

Theorem 14. *While Φ describes $dm(A)$, the identities $\Psi = B(y_1)\pi_Y\Phi$ describe the ideal of polynomial relations, of coefficients in A, among generators of the A-algebra of convergent polyzêtas. Moreover, if the Euler constant, γ, does not belong to A then these relations are algebraically independent on γ.*

Simplyfied computations on Section 3.3 is an example of such identities. Some consequences of Theorem 14 will be drawn in Section 4.2.

3.2.2 Concatenation of Chen generating series

As an example of the action of the differential Galois group of polylogarithms on their asymptotic expansions, we are interrested on the action of their monodromy group which is contained in $\mathrm{Gal}(DE)$.

The monodromies at 0 and 1 of L are given respectively by [35, 32]

$$\mathcal{M}_0\mathrm{L} = \mathrm{L}e^{2\mathrm{i}\pi\mathfrak{m}_0} \quad \text{and} \quad \mathcal{M}_1\mathrm{L} = \mathrm{L}Z_{\sqcup\!\sqcup}^{-1}e^{-2\mathrm{i}\pi x_1}Z_{\sqcup\!\sqcup} = \mathrm{L}e^{2\mathrm{i}\pi\mathfrak{m}_1}, \tag{70}$$

$$\text{where} \quad \mathfrak{m}_0 = x_0 \quad \text{and} \quad \mathfrak{m}_1 = \prod_{l\in\mathcal{L}ynX-X}^{\searrow} e^{-\zeta(\check{S}_l)\,\mathrm{ad}_{S_l}}(-x_1). \tag{71}$$

- If $C = 2\mathrm{i}\pi\mathfrak{m}_0$ then

$$\Phi = Z_{\sqcup\!\sqcup}e^{2\mathrm{i}\pi x_0}, \tag{72}$$

$$\Psi = \exp\Big(\gamma y_1 - \sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}\Big)\pi_Y Z_{\sqcup\!\sqcup} \tag{73}$$

$$= Z_{⧢}. \tag{74}$$

  The monodromy at 0 consists in the multiplication on the right of $Z_{\sqcup\!\sqcup}$ by $e^{2\mathrm{i}\pi x_0}$ and does not modify $Z_{⧢}$.

- If $C = 2\mathrm{i}\pi\mathfrak{m}_1$ then

$$\Phi = e^{-2\mathrm{i}\pi x_1}Z_{\sqcup\!\sqcup}, \tag{75}$$

$$\Psi = \exp\Big(\underbrace{(\gamma - 2\mathrm{i}\pi)}_{T:=}y_1 - \sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}\Big)\pi_Y Z_{\sqcup\!\sqcup} \tag{76}$$

$$= e^{-2\mathrm{i}\pi y_1}Z_{⧢}. \tag{77}$$

  The monodromy at 1 consists in the multiplication on left of $Z_{\sqcup\!\sqcup}$ and of $Z_{⧢}$ by $e^{-2\mathrm{i}\pi x_1}$ and $e^{-2\mathrm{i}\pi y_1}$ respectively.

Remark 1. 1. *The monodromies around singularities of* L *could not allow, in this case, neither to introduce the factor $e^{\gamma x_1}$ on the left of $Z_{\sqcup\!\sqcup}$ nor to eliminate the left factor $e^{\gamma y_1}$ in Z_γ (by putting*[23] *$T = 0$, for example).*

2. *By Proposition 5, we already saw that $Z_{\sqcup\!\sqcup}$ is the concatenation of Chen generating series [10] $e^{x_0\log\varepsilon}$ and then $S_{\varepsilon\rightsquigarrow 1-\varepsilon}$ and finally, $e^{x_1\log\varepsilon}$:*

$$Z_{\sqcup\!\sqcup} \quad \widetilde{_{\varepsilon\to 0^+}} \quad e^{x_1\log\varepsilon}\,S_{\varepsilon\rightsquigarrow 1-\varepsilon}\,e^{x_0\log\varepsilon}. \tag{78}$$

   From (72) and (75), the action of the monodromy group gives

$$e^{x_1\,2k_1\mathrm{i}\pi}Z_{\sqcup\!\sqcup}e^{x_0\,2k_0\mathrm{i}\pi} \quad \widetilde{_{\varepsilon\to 0^+}} \quad e^{x_1(\log\varepsilon+2k_1\mathrm{i}\pi)}\,S_{\varepsilon\rightsquigarrow 1-\varepsilon}\,e^{x_0(\log\varepsilon+2k_0\mathrm{i}\pi)} \tag{79}$$

[23] Why ?

as being the concatenation of the Chen generating series $e^{x_0(\log\varepsilon+2k_0\mathrm{i}\pi)}$ *(along circular path turning* k_0 *times around* 0*), then the Chen generating series* $S_{\varepsilon\rightsquigarrow 1-\varepsilon}$ *and finally, the Chen generating series* $e^{x_1(\log\varepsilon+2k_1\mathrm{i}\pi)}$ *(along circular path turning* k_1 *times around* 1*).*

3. *More generally, by Corollary 7, the action of the Galois differential group of polylogarithms states, for any Lie series* C*, the associator* $\Phi = Z_{⧢}e^C$ *is the concatenation of some Chen generating series* e^C *and* $e^{x_0\log\varepsilon}$ *and then the Chen generating series* $S_{\varepsilon\rightsquigarrow 1-\varepsilon}$ *and finally,* $e^{x_1\log\varepsilon}$ *:*

$$\Phi \underset{\varepsilon\to 0^+}{\widetilde{\qquad}} e^{x_1\log\varepsilon}\, S_{\varepsilon\rightsquigarrow 1-\varepsilon}\, e^{x_0\log\varepsilon}\, e^C. \tag{80}$$

By construction (see Theorem 13) the associator Φ is then the noncommutative generating series of the finite parts of the coefficients of the Chen generating series $S_{z_0\rightsquigarrow 1-z_0}e^C$, for $z_0=\varepsilon\to 0^+$. Hence,

Corollary 9. *Let* $\Phi\in dm(A)$*. For any differential produced formal power series* S *over* X*, there exists*[24] *a* differential representation $(\mathcal{A},f)$ *such that :*

$$\langle\Phi\parallel S\rangle=\sum_{w\in X^*}\langle\Phi\mid w\rangle\,\mathcal{A}(w)\circ f_{|_0}=\prod_{l\in\mathcal{L}ynX-X}^{\searrow} e^{\langle\Phi|\check{S}_l\rangle\ \mathcal{A}(S_l)}\circ f_{|_0}.$$

3.3 Algebraic combinatorial studies of polynomial relation among polyzêta via a group of associators

Here, $\bar{Y}=\{y_1\}\cup\{\bar{y}_k\}_{k\geq 2}$. With the factorization of the monoids X^* and $\bar{Y}^*$ by Lyndon words, let $\{\hat{l}\}_{l\in\mathcal{L}ynX}$ and $\{\hat{l}\}_{l\in\mathcal{L}yn\bar{Y}}$ be the dual of the Lyndon basis over X and $\bar{Y}$.

3.3.1 Preliminary study

As in Definition 10, let

$$A_1=A\epsilon\oplus x_0A\langle X\rangle x_1\quad\text{and}\quad A_2=A\epsilon\oplus(\bar{Y}-\{y_1\})A\langle\bar{Y}\rangle. \tag{81}$$

For $\Phi\in dm(A)$, let $\Psi=B'(y_1)\pi_{\bar{Y}}\Phi$. Let us introduce two algebra morphisms

$$\begin{array}{rclcrcl}\phi:(A_1,⧢)&\longrightarrow&A,&\qquad&\psi:(A_2,\,⧺)&\longrightarrow&A,\\ u&\longmapsto&\langle\Phi\mid u\rangle,&&v&\longmapsto&\langle\Psi\mid v\rangle,\end{array} \tag{82}$$

verifying respectively $\phi(\epsilon)=1,\phi(x_0)=\phi(x_1)=0$ and $\psi(\epsilon)=1,\psi(y_1)=0$.

Lemma 12. *For any* $\Phi\in dm(A)$*, let* $\Psi=B'(y_1)\pi_Y\Phi$*. Then*

$$\begin{array}{rll}&\forall w\in\bar{Y}^*-y_1\bar{Y}^*,&\psi(w)=\phi(\pi_Xw),\\ \textit{or equivalently,}&\forall w\in x_0X^*x_1,&\phi(w)=\psi(\pi_{\bar{Y}}w).\end{array}$$

[24] See Corollary 22 of Annexe B.

Lemma 13. *We have*

$$\Phi = \sum_{u\in X^*} \phi(u)\, u = \prod_{l\in\mathcal{L}ynX-X}^{\searrow} e^{\phi(l)\,\hat{l}} \quad \textit{and} \quad \Psi = \sum_{v\in \bar{Y}^*} \psi(u)\, u = \prod_{l\in\mathcal{L}yn\bar{Y}-\{y_1\}}^{\searrow} e^{\psi(l)\,\hat{l}}.$$

With the notations in Lemma 13, we can state the following

Definition 12. *We put*

$$\mathcal{R} := \bigcap_{\Phi\in dm(A)} \ker\phi \quad (\textit{resp.} \bigcap_{\substack{\Psi=B'(y_1)\pi_{\bar{Y}}\Phi\\ \Phi\in dm(A)}} \ker\psi).$$

Lemma 14. *For any $\Phi \in dm(A)$, let $\Psi = B'(y_1)\pi_{\bar{Y}}\Phi$. Let $Q \in \mathbb{Q}[\mathcal{L}ynX]$ (resp. $\mathbb{Q}[\mathcal{L}yn\bar{Y}]$). Then*

$$\langle Q \parallel \Phi\rangle = 0 \iff Q \in \ker\phi \quad (\textit{resp.} \quad \langle Q \parallel \Psi\rangle = 0 \iff Q \in \ker\psi).$$

Or equivalently (see Definition 7),

$$Q \in \mathcal{R} \quad\iff\quad Q \textit{ is indiscernable over } dm(A).$$

Let $\Phi_1, \Phi_2 \in dm(A)$. By Corollary 7, for $i = 1$ or 2, there exists an unique $P_i \in \mathcal{L}ie_A\langle\langle X\rangle\rangle$ such that e^{-P_i} is well defined and

$$\Phi_i = Z_{\sqcup\!\sqcup} e^{P_i}, \quad \text{or equivalently,} \quad Z_{\sqcup\!\sqcup} = \Phi_1 e^{-P_1} = \Phi_2 e^{-P_2}. \tag{83}$$

Then, we get $\Phi_1 = \Phi_2 e^{P_1-P_2}$ and $\Phi_2 = \Phi_1 e^{P_2-P_1}$. By Lemma 10, it follows

Lemma 15. *Let Φ_1 and $\Phi_2 \in dm(A)$. For any convergent Lyndon word, l, there exists a finite set $I_l \subset \{\lambda \in \mathcal{L}ynX - X$ s.t. $|\lambda| \le |l|\}$ and the coefficients $\{p'_{i,u}\}_{u\in I_l}$ and $\{p''_{i,u}\}_{u\in I_l}$, for $i = 1$ or 2, belonging to A such that*

$$\phi_i(l) = \sum_{u\in I_l} p'_{i,u}\, \zeta(u), \quad \textit{or equivalently,} \quad \zeta(l) = \sum_{u\in I_l} p''_{i,u}\, \phi_i(u).$$

There also exists the coefficients $\{p'_u\}_{u\in I_l}$ and $\{p''_u\}_{u\in I_l}$ belonging to A such that

$$\phi_1(l) = \sum_{u\in I_l} p'_u\, \phi_2(u), \quad \textit{or equivalently,} \quad \phi_2(l) = \sum_{u\in I_l} p''_u\, \phi_1(u).$$

Therefore, the $\{\phi_i(l)\}_{l\in\mathcal{L}ynX-X}$ (resp. $\{\psi_i(l)\}_{l\in\mathcal{L}yn\bar{Y}-\{y_1\}}$), for $i = 1$ or 2, are also generators of the A-algebra generated by convergent polyzêtas.

3.3.2 Description of polynomial relations among coefficients of associator and irreducible polyzêtas

Since the identities of Corollary 8 (see also Corollary 6) hold for any pair of bases, in duality, compatible with factorization of the monoid X^* (resp. $\bar{Y}^*$) then, by Corollary 8, one gets

Theorem 15. *For any $\Phi \in dm(A)$, let $\Psi = B'(y_1)\pi_{\bar{Y}}\Phi$. We have*

$$\prod_{l \in \mathcal{L}yn\bar{Y}-y_1}^{\searrow} e^{\psi(l)\,\hat{l}} \quad = \quad \exp\Big(\sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}\Big)\pi_{\bar{Y}} \prod_{l \in \mathcal{L}ynX-X}^{\searrow} e^{\phi(l)\,\hat{l}}.$$

If $\Phi = Z_{\sqcup\!\sqcup}$ and $\Psi = Z_{\pm\!\!\!\!\sqcup\!\sqcup}$ then, for $\ell \in \mathcal{L}ynX - X$ (resp. $\mathcal{L}yn\bar{Y} - y_1$), one has $\zeta(l) = \phi(l)$ (resp. $\psi(l)$). Hence, one ontains (see also Corollary 6)

Theorem 16 (Bis repetita)**.**

$$\prod_{l \in \mathcal{L}yn\bar{Y}-y_1}^{\searrow} e^{\zeta(l)\,\hat{l}} \quad = \quad \exp\Big(\sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}\Big)\pi_{\bar{Y}} \prod_{l \in \mathcal{L}ynX-X}^{\searrow} e^{\zeta(l)\,\hat{l}}.$$

Corollary 10. *For any $\ell \in \mathcal{L}yn\bar{Y} - y_1$ (resp. $\mathcal{L}ynX - X$), let $P_\ell \in \mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X\rangle)$ (resp. $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle \bar{Y}\rangle)$) be the decomposition of the polynomial $\pi_X\hat{\ell} \in \mathbb{Q}\langle X\rangle$ (resp. $\pi_{\bar{Y}}\hat{\ell} \in \mathbb{Q}\langle \bar{Y}\rangle$) in the PBW basis, induced by $\{\hat{l}\}_{l\in\mathcal{L}ynX}$ (resp. $\{\hat{l}\}_{l\in\mathcal{L}yn\bar{Y}}$), and let $\check{P}_\ell \in \mathbb{Q}[\mathcal{L}ynX - X]$ (resp. $\mathbb{Q}[\mathcal{L}yn\bar{Y} - y_1]$) be its dual. Then one obtains*

$$\pi_X\ell - \check{P}_\ell \in \ker\phi \quad (\textit{resp.} \quad \pi_{\bar{Y}}\ell - \check{P}_\ell \in \ker\psi).$$

In particular, for $\phi = \zeta$ (resp. $\psi = \zeta$) then one also obtains

$$\pi_X\ell - \check{P}_\ell \in \ker\zeta \quad (\textit{resp.} \quad \pi_{\bar{Y}}\ell - \check{P}_\ell \in \ker\zeta).$$

Moreover, for any $\ell \in \mathcal{L}yn\bar{Y} - y_1$ (resp. $\mathcal{L}ynX - X$), the homogenous polynomial $\pi_X\ell - \check{P}_\ell \in \mathbb{Q}\langle X\rangle$ (resp. $\mathbb{Q}\langle \bar{Y}\rangle$) is of degree equal $\mid \ell \mid \geq 2$.

Proof. Since

$$\ell \in \mathcal{L}yn\bar{Y} \quad \Longleftrightarrow \quad \pi_X\ell \in \mathcal{L}ynX - \{x_0\}$$

then identifying the local coordinates (of second kind) on the two members of each identity in Theorem 15, one obtains

$$\begin{aligned} \forall \ell \in \mathcal{L}yn\bar{Y} - y_1 \subset Y^* - y_1Y^*, &\qquad \psi(\ell) = \phi(\check{P}_\ell), \\ (\text{resp. } \forall \ell \in \mathcal{L}ynX - X \subset x_0X^*x_1, &\qquad \phi(\ell) = \psi(\check{P}_\ell)). \end{aligned}$$

By Lemma 12, we get the expected result. □

With the notations of Corollary 10, we get the following

Definition 13. *Let Q_ℓ be the decomposition of the proper polynomial $\pi_{\bar{Y}}\ell - \check{P}_\ell$ (resp. $\pi_X\ell - \check{P}_\ell$) in $\mathcal{L}yn\bar{Y}$ (resp. $\mathcal{L}ynX$). Let*

$$\begin{aligned} \mathcal{R}_{\bar{Y}} := \{Q_\ell\}_{\ell\in\mathcal{L}yn\bar{Y}-y_1} \quad &\textit{and} \quad \mathcal{R}_X := \{Q_\ell\}_{\ell\in\mathcal{L}ynX-X}, \\ \mathcal{L}_{irr}\bar{Y} := \{\ell \in \mathcal{L}yn\bar{Y} - y_1 \mid Q_\ell = 0\} \quad &\textit{and} \quad \mathcal{L}_{irr}X := \{\ell \in \mathcal{L}ynX - X \mid Q_\ell = 0\}. \end{aligned}$$

It follows that

Lemma 16. *We have*

$$\begin{array}{rcl}(\mathbb{Q}[\mathcal{L}yn\bar{Y}-y_1],\ \text{ш}) & = & (\mathcal{R}_{\bar{Y}},\ \text{ш})\oplus(\mathbb{Q}[\mathcal{L}_{irr}\bar{Y}],\ \text{ш}),\\ (\mathbb{Q}[\mathcal{L}ynX-X],\sqcup\!\sqcup) & = & (\mathcal{R}_X,\sqcup\!\sqcup)\oplus(\mathbb{Q}[\mathcal{L}_{irr}X],\sqcup\!\sqcup).\end{array}$$

Then we can state the following

Definition 14. *Any word w is said to be* irreducible *if and only if w belongs to $\mathcal{L}_{irr}\bar{Y}$ (resp. $\mathcal{L}_{irr}X$). In this case, the polyzêta $\zeta(w)$ est said to be* $\mathbb{Q}$-irreducible.

For any $P\in\mathbb{Q}[\mathcal{L}_{irr}X]$, there exists[25] a *differential representation* $(\mathcal{A},f)$ such that P can be *finitely* factorized (see also Corollary 9) :

$$P=\sigma f_{|_0}=\sum_{w\in X^*_{irr}}\mathcal{A}(w)\circ f\ w=\prod^{\searrow}_{\ell\in\mathcal{L}_{irr}X,\text{finite}}e^{\mathcal{A}(\hat{\ell})\ \ell}\circ f, \tag{84}$$

where X^*_{irr} denotes the set of words obtaining by shuffling on $\mathcal{L}_{irr}X$.

Lemma 17. *Any proper polynomial $P\in(\mathbb{Q}[\mathcal{L}_{irr}X],\sqcup\!\sqcup)$ (resp. $(\mathbb{Q}[\mathcal{L}_{irr}\bar{Y}],\ \text{ш})$) is indiscernable over Chen generating series $\{e^{t\ x}\}^{t\in\mathbb{C}}_{x\in X}$:*

$$\langle P\parallel e^{t\ x_0}\rangle=\langle P\parallel e^{t\ x_1}\rangle=0\quad(\textit{resp.}\quad\langle P\parallel e^{ty_1}\rangle=0).$$

Proof. By construction, x_0 and $x_1\notin\mathcal{L}_{irr}X$ (resp. $y_1\notin\mathcal{L}_{irr}X$). For any $n>1, x_0^n$ and x_1^n (resp. y_1^n) are not Lyndon words then they do not belong to $\mathcal{L}_{irr}X$ (resp. $\mathcal{L}_{irr}X$). Therefore, for any $n\ge0$, one has $\langle P\mid x_0^n\rangle=\langle P\mid x_1^n\rangle=0$ (resp. $\langle P\mid y_1^n\rangle=0$). Using the expansion of the exponential, we find the expected result. □

Lemma 18. *Let $\Phi\in dm(A)$ and let $t\in\mathbb{C},x\in X$. For any proper polynomial $P\in(\mathbb{Q}[\mathcal{L}_{irr}X],\sqcup\!\sqcup)$, if $\langle P\parallel\Phi\rangle=0$ then $\langle P\parallel\Phi e^{t\ x}\rangle=0$ and $\langle P\parallel e^{t\ x}\Phi\rangle=0$.*

Proof. Since $P\in(\mathbb{Q}[\mathcal{L}_{irr}X],\sqcup\!\sqcup)$ and P is proper then, by Lemma 17, for any $t\in\mathbb{C}$ and for any $x\in X$, we have $\langle P\parallel e^{t\ x}\rangle=0$ and then $\langle P\parallel\Phi e^{t\ x}\rangle=0$.

Since $\mathrm{supp}(P)\subset x_0X^*x_1$ then $\langle P\parallel e^{t\ x_0}\Phi\rangle=\langle P\triangleright e^{t\ x_0}\parallel\Phi\rangle=0$.

Next, for $\Phi\in dm(A)$, there exists e^C such that $e^{t\ x_1}\Phi=e^{t\ x_1}Z_{\sqcup\!\sqcup}e^C$ and, by Proposition 5, we get

$$e^{t\ x_1}\Phi\ \widetilde{_{\varepsilon\to0^+}}\ e^{x_1(t+\log\varepsilon)}\ S_{\varepsilon\rightsquigarrow1-\varepsilon}\ e^{x_0\log\varepsilon}\ e^C.$$

Hence, there exists a Chen generating series $C_{z\rightsquigarrow1-z_0}$ and $S_{z_0\rightsquigarrow1-z_0}$ such that we get the following asymptotic behaviour (see Section 3.2.2)

$$e^{t\ x_1}\Phi\ \widetilde{_{\varepsilon\to0^+}}\ C_{z\rightsquigarrow1-z_0}S_{z_0\rightsquigarrow z}\ e^C$$

and the following concatenation holds [10] (see Formula (53))

$$\begin{array}{rrcl} & C_{z\rightsquigarrow1-z_0}S_{z_0\rightsquigarrow z} & = & S_{z_0\rightsquigarrow1-z_0},\\ \iff & C_{z\rightsquigarrow1-z_0}S_{z_0\rightsquigarrow z}e^C & = & S_{z_0\rightsquigarrow1-z_0}e^C.\end{array}$$

[25] See Corollary 22 of Annexe B.

Since $P \in \mathbb{Q}[\mathcal{L}_{irr}X]$ then by (84), applying $\langle \sigma f_{|_0} \parallel \bullet \rangle$ to the two sides of the previous equality, one has

$$\langle \sigma f_{|_0} \parallel C_{z \rightsquigarrow 1-z_0} S_{z_0 \rightsquigarrow z} e^C \rangle \quad = \quad \langle \sigma f_{|_0} \parallel S_{z_0 \rightsquigarrow 1-z_0} e^C \rangle.$$

Thus, for $z_0 = \varepsilon \to 0^+$, one obtains

$$\langle \sigma f_{|_0} \parallel e^{t\, x_1} \Phi \rangle \underset{\varepsilon \to 0^+}{\widetilde{\quad}} \langle \sigma f_{|_0} \parallel \Phi \rangle.$$

Since $\langle \sigma f_{|_0} \parallel \Phi \rangle = \langle P \parallel \Phi \rangle = 0$ then we get the expected result. □

Lemma 19. *For any $\Phi \in dm(A)$, let $\Psi = B'(y_1)\pi_{\bar{Y}}\Phi$. We have $\mathcal{R}_{\bar{Y}} \subseteq \ker \psi$ and $\mathcal{R}_X \subseteq \ker \phi$. In particular, $\mathcal{R}_{\bar{Y}} \subseteq \ker \zeta$ and $\mathcal{R}_X \subseteq \ker \zeta$.*

Proposition 13. *We have $\mathcal{R}_X \subseteq \mathcal{R}$ (resp. $\mathcal{R}_{\bar{Y}} \subseteq \mathcal{R}$).*

Proposition 14. *For any proper polynomial $Q \in \mathbb{Q}[\mathcal{L}_{irr}X]$ (resp. $\mathbb{Q}[\mathcal{L}_{irr}\bar{Y}]$),*

$$Q \in \mathcal{R} \quad \Longleftrightarrow \quad Q = 0.$$

Proof. If $Q = 0$ then since, for $\Phi \in dm(A), \phi$ is an algebra homorphism then $\phi(Q) = 0$. Hence, $Q \in \ker \phi$ and then $Q \in \mathcal{R}$.

Conversely, if $Q \in \mathcal{R}$ then, for $\Phi \in dm(A)$, we get $\langle Q \parallel \Phi \rangle = 0$. That means Q is indiscernable over $dm(A)$. Let $\mathcal{H}$ be the monoid generated by $dm(A)$ and by the Chen generating series $\{e^{t\, x}\}_{x \in X}^{t \in \mathbb{C}}$. By Lemma 25, Q is continuous over $\mathcal{H}$ and by Lemma 18, it is indiscernable over $\mathcal{H}$. By Proposition 7, the expected result follows. □

Therefore, by the propositions 13 and 14, we obtain

Theorem 17. *We have $\mathcal{R} = \mathcal{R}_X$ (resp. $\mathcal{R}_{\bar{Y}}$).*

Proposition 15. *For any $\Phi \in dm(A)$, let $\Psi = B'(y_1)\pi_{\bar{Y}}\Phi$. Let $Q \in (\mathbb{Q}[\mathcal{L}_{irr}X], \sqcup\!\sqcup)$ (resp. $(\mathbb{Q}[\mathcal{L}_{irr}\bar{Y}], \uplus)$) such that $\langle \Phi \parallel Q \rangle = 0$ (resp. $\langle \Psi \parallel Q \rangle = 0$). Then $Q = 0$.*

Proof. Let $\mathcal{H}$ defined as being the monoid generated by Φ and by Chen generating series $\{e^{t\, x}\}_{x \in X}^{t \in \mathbb{C}}$. By assumption, $\langle \Phi \parallel Q \rangle = 0$ and by Lemma 18, Q is then indiscernable over $\mathcal{H}$. Finally, by Proposition 7, it follows that $Q = 0$. □

Proposition 16. *For any $\Phi \in dm(A)$, let $\Psi = B'(y_1)\pi_{\bar{Y}}\Phi$. We get $\ker \phi = \mathcal{R}_X$ (resp. $\ker \psi = \mathcal{R}_{\bar{Y}}$). In particular, $\ker \zeta = \mathcal{R}_X$ (resp. $\ker \zeta = \mathcal{R}_{\bar{Y}}$).*

Proof. By Lemma 19, $\mathcal{R}_X$ and $\mathcal{R}_{\bar{Y}}$ are included in $\ker \phi$ and $\ker \psi$ respectively.

Conversely, two cases can occur (see Lemma 16) :

1. Case $Q \notin \mathbb{Q}[\mathcal{L}_{irr}X]$ (resp. $\mathbb{Q}[\mathcal{L}_{irr}\bar{Y}]$). By Lemma 16, $Q \equiv_{\mathcal{R}_X} Q_1$ (resp. $Q \equiv_{\mathcal{R}_{\bar{Y}}} Q_1$) such that $Q_1 \in \mathbb{Q}[\mathcal{L}_{irr}X]$ (resp. $\mathbb{Q}[\mathcal{L}_{irr}\bar{Y}]$) and $\phi(Q_1) = 0$ (resp. $\psi(Q_1) = 0$). This case is then reduced to the following
2. Case $Q \in \mathbb{Q}[\mathcal{L}_{irr}X]$ (resp. $\mathbb{Q}[\mathcal{L}_{irr}\bar{Y}]$). Using Proposition 15, we have $Q \equiv_{\mathcal{R}_X} 0$ (resp. $Q \equiv_{\mathcal{R}_{\bar{Y}}} 0$).

Then, $\mathcal{R}_X$ (resp. $\mathcal{R}_{\bar{Y}}$) contains $\ker\phi$ (resp. $\ker\psi$). $\square$

For any $Q \in (\mathbb{Q}[\mathcal{L}_{irr}X], ⧢)$ (resp. $(\mathbb{Q}[\mathcal{L}_{irr}\bar{Y}], \uplus))$, $\zeta(Q)$ is then a polynomial on A-irreducible polyzêtas (see Definition 14). Moreover,

Proposition 17. *The $\mathbb{Q}$-algebra $\mathcal{Z}$ is generated by the family of A-irreducible polyzêtas $\{\zeta(\ell)\}_{\ell\in\mathcal{L}_{irr}\bar{Y}}$ (resp. $\{\zeta(\ell)\}_{\ell\in\mathcal{L}_{irr}X}$).*

Proof. By Radford's theorem [45], one just needs to prove for Lyndon words :

Let $\ell \in \mathcal{L}yn\bar{Y}-y_1$. If $\pi_X\ell = \check{P}_\ell$ then the result follows else one has $\pi_X\ell-\check{P}_\ell \in \ker\zeta$. Hence, $\zeta(\ell) = \zeta(\check{P}_\ell)$.

Since $\check{P}_\ell \in \mathbb{Q}[\mathcal{L}ynX - X]$ then $\check{P}_\ell$ is polynomial on Lyndon words, over X, of degree less or equal $\mid \ell \mid$. For each Lyndon word does appear in this decomposition of $\check{P}_\ell$, after applying $\pi_{\bar{Y}}$, one uses the same recurcive procedure until getting Lyndon words in $\mathcal{L}_{irr}\bar{Y}$.

The same treatment works for any $\ell' \in \mathcal{L}ynX - X$. $\square$

For any $\Phi \in dm(A)$, by Proposition 16, one also has $\ker\phi = \ker\zeta = \mathcal{R}_X$. That means, for any irreducible Lyndon words $l \neq l'$,

$$\phi(l) = \phi(l') \quad\Longleftrightarrow\quad \zeta(l) = \zeta(l'). \tag{85}$$

Let us state then the following

Lemma 20. *Let $\Phi \in dm(A)$. Let us define the map*

$$\varphi : \mathcal{Z} \longrightarrow A$$

as follows

$$\forall l \in \mathcal{L}_{irr}X, \qquad \varphi(\zeta(l)) := \phi(l).$$

Then φ is an algebra homomorphism and $\{\varphi(\zeta(l))\}_{l\in\mathcal{L}_{irr}X}$ are generators of A.

Thus, for any $\theta \in \mathcal{Z}$ there exist the coefficients $\{\alpha_{l_1,\ldots,l_n}\}^{n\in\mathbb{N}}_{l_1,\ldots,l_n\in\mathcal{L}_{irr}X}$ in A such that (see Proposition 17 and Lemme 20)

$$\varphi(\theta) \quad=\quad \sum_{n\geq 0}\sum_{l_1,\ldots,l_n\in\mathcal{L}_{irr}X} \alpha_{l_1,\ldots,l_n}\ \varphi(\zeta(l_1))\ldots\varphi(\zeta(l_n)). \tag{86}$$

In particular, since for any $w \in X^*$, $\zeta_⧢(w)$ belongs to $\mathcal{Z}$ (see Corollary 3) then $\varphi(\zeta_⧢(w))$ is well defined and $\varphi(\zeta_⧢(w))$ can be expressed as polynomial on convergent polyzêtas with coefficients in A :

Lemma 21. *With the notations in Lemma 20, one has*

$$\forall w \in X^*, \quad \varphi(\zeta_⧢(w)) \quad=\quad \sum_{\substack{u,v\in X^*\\ uv=w}} \langle e^C \mid v\rangle\ \zeta_⧢(u).$$

Proof. The expected result follows by identifying coefficients in $\Phi = Z_⧢ e^C$. $\square$

Finally, we can state the following

Theorem 18. *For any $\Phi \in dm(A)$, there exists an unique algebra homomorphism*

$$\varphi : \mathcal{Z} \longrightarrow A$$

such that Φ is computed from $Z_{\sqcup\!\sqcup}$ by applying φ to each coefficient :

$$\Phi = \sum_{w \in X^*} \varphi(\zeta_{\sqcup\!\sqcup}(w))\, w = \prod_{l \in \mathcal{L}ynX - X}^{\searrow} e^{\varphi(\zeta(l))\, \hat{l}}.$$

Remark 2. *1. In this work, neither the question deciding any real number belongs to $\mathcal{Z}$ or not nor the question expliciting $\{\alpha_{l_1,\ldots,l_n}\}_{l_1,\ldots,l_n \in \mathcal{L}_{irr}X}^{n \in \mathbb{N}}$ in (86), are considered.*

2. Now, by considering the commutative indeterminates $t_1, t_2, t_3, \ldots$, let A be the $\mathbb{Q}$-algebra obtained by specializing $\mathbb{Q}[t_1, t_2, t_3, \ldots]$ at $t_1 = \mathrm{i}\pi$:

$$A = \mathbb{Q}[\mathrm{i}\pi][t_2, t_3, \ldots]. \tag{87}$$

Neither the Lie exponential series $\mathrm{e}^{\mathrm{i}\pi x_0}$ nor $\mathrm{e}^{\mathrm{i}\pi x_1}$ does belong to $dm(A)$ but it belongs to $\mathrm{Gal}(DE)$. In particular, it figures in the modromies (see Section 3.2.2) or in the functional relations (see (45) and (46)) of polylogarithms and in the hexagonal relation of polyzêtas (see Proposition 6).

3. Applying Baker–Campbell–Hausdorff formula [6] to Proposition 6 we get, at orders 2 and 3 as examples, the famous Euler's formula saying $\zeta(2)$ is an algebraic number over $A = \mathbb{Q}[\mathrm{i}\pi]$:

$$\zeta(2) + \frac{(\mathrm{i}\pi)^2}{6} = 0 \quad \textit{(order 2)}, \tag{88}$$

$$\zeta(3) - \zeta(2,1) = 0 \quad \textit{(order 3, imaginary part)}. \tag{89}$$

Therefore, the first comming in mind homomorphism

$$\varphi : \mathcal{Z} \longrightarrow A$$

maps, at least $\zeta(2)$ to $\varphi(\zeta(2)) = \pi^2/6$.

4. For this reason, in [30], we have to consider the $\mathbb{Q}$-algebra generated by $\mathrm{i}\pi$ and by other A-irreducible polyzêtas obtained in [32, 34, 3, 49] (and such algebra is denoted in this work by A). This algebra came up from the studies of monodromies [32, 35], as already shown in (70), and the Kummer type functional equations of polylogarithms [32, 36], as already shown in (44)–(46). In particular, by (46), we get for example [36, 32],

$$\begin{aligned} \mathrm{Li}_{2,1} \frac{1}{t} &= -\frac{(\mathrm{i}\pi)^2}{2} \log t + \mathrm{i}\pi(\zeta(2) - \frac{\log^2 t}{2} - \mathrm{Li}_2\, t) \\ &\quad - \mathrm{Li}_{2,1}\, t + \mathrm{Li}_3\, t - \log t\, \mathrm{Li}_2\, t + \zeta(3) - \frac{\log^3 t}{6}. \end{aligned} \tag{90}$$

specializing $t = 1$, the real part of this leads again to the Euler's indentity (89).

4 Concluding remarks : complete description of $\ker\zeta$ and structure of polyzêtas

For the same raison as already said in Remark 2(2), let us consider now[26] the commutative indeterminates $t_1, t_2, t_3, \ldots$.

Let A be the $\mathbb{Q}$-algebra obtained by specializing $\mathbb{Q}[t_1, t_2, t_3, \ldots]$ at $t_1 = \mathrm{i}\pi$:

$$A = \mathbb{Q}[\mathrm{i}\pi][t_2, t_3, \ldots]. \tag{91}$$

4.1 A conjecture by Pierre Cartier

Definition 15 ([8, 43])**.** *Let $DM(A)$ denotes the set of $\Phi \in A\langle\langle X\rangle\rangle$ such that*

$$\langle \Phi \mid \epsilon\rangle = 1, \quad \langle \Phi \mid x_0\rangle = \langle \Phi \mid x_1\rangle = 0, \quad \Delta_{⧢}\Phi = \Phi \otimes \Phi$$

and such that, for

$$\bar{\Psi} = \exp\biggl(-\sum_{n\geq 2}\langle \pi_Y\Phi \mid y_n\rangle \frac{(-y_1)^n}{n}\biggr)\pi_Y\Phi \in A\langle\langle Y\rangle\rangle,$$

then $\Delta_{\boxplus}\bar{\Psi} = \bar{\Psi}\otimes\bar{\Psi}$.

Since $DM(A)$ contains already $Z_{⧢}$ then for $\Phi \in DM(A)$, by Theorem 13, there exists $C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle$ verifying

$$\langle e^C \mid \epsilon\rangle = 1 \quad \text{and} \quad \langle e^C \mid x_0\rangle = \langle e^C \mid x_1\rangle = 0$$

such that

$$\Phi \quad = \quad Z_{⧢} e^C \tag{92}$$

and such that

$$\Psi = B'(y_1)\pi_Y\Phi = \exp\biggl(-\sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}\biggr)\pi_Y\Phi, \tag{93}$$

$$\bar{\Psi} = \exp\biggl(-\sum_{n\geq 2}\langle \pi_Y\Phi \mid y_n\rangle \frac{(-y_1)^n}{n}\biggr)\pi_Y\Phi. \tag{94}$$

By construction (see Definition 11 and Theorem 13), such Φ and Ψ are group-like (for the co-products $\Delta_{⧢}$ and $\Delta_{\boxplus}$ respectively) and here, $\bar{\Psi}$ must be also group-like (for the co-product $\Delta_{\boxplus}$). If such a Lie series C exists then it is unique, due to the fact that $e^C = \Phi Z_{⧢}^{-1}$, and it is group-like (for the co-product $\Delta_{⧢}$).

[26]We do not consider in any case $A = \mathbb{Q}$ as in previous versions.

Corollary 11 (conjectured by Cartier, [8]). *For any $\Phi \in DM(A)$, there exists an unique algebra homomorphism*[27]

$$\bar{\varphi} : \mathcal{Z} \longrightarrow A$$

such that Φ is computed from $Z_{\sqcup\!\sqcup}$ by applying $\bar{\varphi}$ to each coefficient.

Proof. By Theorem 18, use the fact

$$DM(\mathbb{Q}) \subseteq DM(A) \subseteq dm(A).$$

$\square$

4.2 Arithmetical nature of γ

By Theorem 14, under the assumption that the Euler constant, γ, does not belong to a commutative $\mathbb{Q}$-algebra A then γ does not verify any polynomial with coefficients in A among the convergent polyzêtas. It follows then,

Corollary 12. *If $\gamma \notin A$ then it is transcendental over the A-algebra generated by the convergent polyzêtas.*

Or equivalently, by contraposition,

Corollary 13. *If there exists a polynomial relation with coefficients in A among the Euler constant, γ, and the convergent polyzêtas then $\gamma \in A$.*

Therefore,

Corollary 14. *If the Euler constant, γ, does not belong to A then γ is not algebraic over A.*

Remark 3. *1. In the same spirit of Theorem 11, let $\zeta^T_{\boxplus}$ be the regularization morphism*[28] *from $(\mathbb{Q}\langle Y\rangle, \boxplus)$ to $(\mathbb{R}, .)$ mapping y_1 to T. Let $Z^T_{\boxplus}$ be the noncommutative generating series of polyzêtas regularisized with respect to $\zeta^T_{\boxplus}$. Thus, as in Theorem 11 and by infinite factorization by Lyndon words, we also get*

$$Z^T_{\boxplus} := \sum_{w \in X^*} \zeta^T_{\boxplus}(w)\, w = e^{Ty_1} Z_{\boxplus}. \tag{95}$$

2. Now let us consider the regularization, for $N \to +\infty$ and with respect to $\zeta^T_{\boxplus}$, of the power series Const(N) *given in (50) as*

$$B^T(y_1) \quad = \quad e^{Ty_1} B'(y_1) \tag{96}$$

[27] See Remark 2(3) to have an example of $\bar{\varphi}$.

[28] This is a *symbolic* regularization and does not yet have an analytical justification as it is done, separately, for $\zeta_{\sqcup\!\sqcup}$ and $\zeta_{\boxplus}$ in Section 3.1.2 as finite parts of the asymptotic expansions, in different scales of comparison, of $\mathrm{Li}_{x_1}(z)$, for $z \rightsquigarrow 1$, and $\mathrm{H}_{y_1}(N)$, for $N \to \infty$, respectively.

As in Corollary 6, we always get

$$Z_{\text{⧢}}^T = B^T(y_1)\pi_Y Z_{\sqcup\!\sqcup} \quad\Longleftrightarrow\quad Z_{\text{⧢}} = B'(y_1)\pi_Y Z_{\sqcup\!\sqcup}. \tag{97}$$

Hence, roughly speaking, for the quasi-shuffle product, the symbolic regularization to T is also "equivalent" to the regularization to 0.

3. *Again, as in Corollary 12, if $T \notin A$ then $T \notin \bar{A}$.*

   A contrario, as in Corollary 13, if there exists a polynomial relation with coefficients in A among T and convergent polyzêtas then $T \in A$.

4.3 Structure and arithmetical nature of polyzêtas

Once again, let us consider

$$\begin{aligned}
(A_1, \sqcup\!\sqcup) &= (A\epsilon \oplus x_0 A\langle X\rangle x_1, \sqcup\!\sqcup) && (98)\\
&\cong A[\mathcal{L}ynX - X], \sqcup\!\sqcup) && (99)\\
&= (\mathcal{R}_X, \sqcup\!\sqcup) \oplus (A[\mathcal{L}_{irr}X], \sqcup\!\sqcup), && (100)\\
(A_2, \text{⧢}) &= (A\epsilon \oplus (\bar{Y} - \{y_1\})A\langle \bar{Y}\rangle, \text{⧢}) && (101)\\
&\cong (A[\mathcal{L}yn\bar{Y} - y_1], \text{⧢}) && (102)\\
&= (\mathcal{R}_{\bar{Y}}, \text{⧢}) \oplus (A[\mathcal{L}_{irr}\bar{Y}], \text{⧢}) && (103)
\end{aligned}$$

(see Definition 10, Lemma 16, Definition 13). Then $(A_1, \sqcup\!\sqcup) \cong (A_2, \text{⧢})$ [32, 33].

Let us consider again the following algebra morphism (see Proposition 9)

$$\begin{aligned}
\zeta \quad : \quad \begin{matrix}(A_2, \text{⧢})\\ (A_1, \sqcup\!\sqcup)\end{matrix} &\longrightarrow (\mathbb{R}, .) && (104)\\
\begin{matrix} y_{r_1}\ldots y_{r_k}\\ x_0x_1^{r_1-1}\ldots x_0x_1^{r_k-1}\end{matrix} &\longmapsto \sum_{n_1>\ldots>n_k>0} \frac{1}{n_1^{r_1}\ldots n_k^{r_k}}. && (105)
\end{aligned}$$

Lemma 22. *The image of the algebra morphism ζ is $\mathcal{Z}$.*

Let us make precise the structure of $\mathcal{Z}$ and the arithmetical nature of polyzêtas :

As consequences of the propositions 15, 16 and 17, by taking $\Phi = Z_{\sqcup\!\sqcup}$, we have

$$\begin{aligned}
\operatorname{Im}\zeta = \zeta(\mathrm{A}[\mathcal{L}_{irr}\bar{Y}]) \quad &\text{and} \quad \ker\zeta = \mathcal{R}_{\bar{Y}} && (106)\\
(\text{resp.}\quad \operatorname{Im}\zeta = \zeta(\mathrm{A}[\mathcal{L}_{irr}X]) \quad &\text{and} \quad \ker\zeta = \mathcal{R}_X). && (107)
\end{aligned}$$

By Corollary 10, $\ker\zeta$ is an ideal generated by the homogenous polynomials of degree ≥ 2. Hence, the quotien $A_1/\mathcal{R}_X$ or $A_2/\mathcal{R}_{\bar{Y}}$ (the source by the kernel of ζ) is graded [6] and it is isomorphic to $\mathrm{Im}\zeta$.

Therefore, by Lemma 22 and Proposition 17, we obtain respectively the following direct consequences

Theorem 19 (Structure of polyzêtas)**.** *The A-algebra $\mathcal{Z}$ is*

1. *isomorphic to the graded algebra* $(A_1/\mathcal{R}_X, ⧢)$*, or equivalently,* $(A_2/\mathcal{R}_{\bar{Y}}, \uplus)$*.*

2. *freely generated by the A-irreducible polyzêtas* $\{\zeta(l)\}_{l\in\mathcal{L}_{irr}\bar{Y}}$ *(resp.* $\{\zeta(l)\}_{l\in\mathcal{L}_{irr}X}$*).*

For any $p \geq 2$, let

$$\mathcal{Z}_p \quad = \quad \mathrm{span}_{\mathbb{Q}}\{\zeta(w) \mid w \in x_0X^*x_1, |w| = p\}. \tag{108}$$

By definition of graded algebra [6], Theorem 19 means also that

$$\mathcal{Z} \quad = \quad A \oplus \bigoplus_{p\geq 2} \mathcal{Z}_p \tag{109}$$

and there is no linear relation among elements of different $\mathcal{Z}_p$ ([8] conjecture C1, [48]).

Thus, if θ is a (A-irreducible) polyzêta verifying the following algebraic equation

$$\theta^n + a_{n-1}\theta^{n-1} + \ldots + a_0 \quad = \quad 0 \tag{110}$$

then $\theta = 0$ because $\mathcal{Z}_{p_1}\mathcal{Z}_{p_2} \subset \mathcal{Z}_{p_1+p_2}$, for $p_1, p_2 \geq 2$, and each monomial in (110) is then of different weight. By consequence,

Corollary 15. *Any (A-irreducible) polyzêta* θ *is a transcendental over* $\mathbb{Q}$*.*

Remark 4. *In this work, neither the study of* $\dim \mathcal{Z}_p$ *[50] (see also [8], conjecture C2) nor the estimate of the number of A-irreducible polyzêtas generating* $\mathcal{Z}_p$*, are discussed knowing the A-irreducible polyzêtas form transcendence basis of the A-algebra* $\mathcal{Z}$*.*

5 Annexe A : pair of bases in duality and proof of Theorem 2

5.1 Preliminary results

Let $\mathbb{Q}\langle Y\rangle$ be equipped the concatenation and the quasi-shuffle, $\uplus$, defined by

$$\forall y_i, y_j \in Y = \{y_i\}_{i\geq 1}, \forall u, v \in Y^*, \qquad y_iu \uplus y_jv = y_i(u \uplus y_jv) + y_{i+j}(y_iu \uplus v),$$
$$\forall w \in Y^*, \qquad w \uplus 1_{Y^*} = 1_{Y^*} \uplus w = w,$$

or by its associated co-product, $\Delta_{\uplus}$, defined by

$$\forall y_k \in Y, \qquad \Delta_{\uplus}(y_k) = y_k \otimes 1 + 1 \otimes y_k + \sum_{i+j=k} y_i \otimes y_j.$$

satistying, for any $u, v, w \in Y^*$, $\langle u \otimes v \mid \Delta_{\uplus}(w)\rangle = \langle u \uplus v \mid w\rangle$.

Lemma 23. *Let* $S_1, \ldots, S_n$ *be* proper *formal power series in* $\mathbb{Q}\langle\langle Y\rangle\rangle$*. Let* $P_1, \ldots, P_m$ *be primitive elements*[29] *in* $\mathbb{Q}\langle Y\rangle$*, for the co-product* $\uplus$*.*

[29] *i.e.*, for any $i = 1, .., m, \Delta_{\uplus}(P_i) = 1 \otimes P_i + P_i \otimes 1$.

1. *If $n > m$ then $\langle S_1 \uplus \ldots \uplus S_n \mid P_1 \ldots P_m \rangle = 0$.*
2. *If $n = m$ then*

$$\langle S_1 \uplus \ldots \uplus S_n \mid P_1 \ldots P_n \rangle = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} \langle S_i \mid P_{\sigma(i)} \rangle.$$

3. *If $n < m$ then, by considering the language $\mathcal{M}$ over $\mathcal{A} = \{P_1, \ldots, P_m\}$*

$$\mathcal{M} = \{w \in \mathcal{A}^* | w = P_{j_1} \ldots P_{j_{|w|}}, j_1 < \ldots < j_{|w|}, |w| \geq 1\}$$

and the morphism $\mu : \mathbb{Q}\langle \mathcal{A} \rangle \longrightarrow \mathbb{Q}\langle Y \rangle$, one has :

$$\langle S_1 \uplus \ldots \uplus S_n \mid P_1 \ldots P_m \rangle = \sum_{\substack{w_1, \ldots, w_m \in \mathcal{M} \\ |w_1| + \ldots + |w_n| = m \\ \forall i,j=1,..m, \mathrm{alp}(w_i) \cap \mathrm{alp}(w_j) = \emptyset}} \prod_{i=1}^{n} \langle S_i \mid \mu(w_i) \rangle.$$

Proof. On one hand, since the P_i's are primitive then

$$\Delta_{\uplus}^{(n-1)}(P_i) = \sum_{p+q=n-1} 1^{\otimes p} \otimes P_i \otimes 1^{\otimes q}.$$

On the other hand, $\langle S_1 \uplus \ldots \uplus S_n \mid P_1 \ldots P_m \rangle = \langle S_1 \otimes \ldots \otimes S_n \mid \Delta_{\uplus}^{(n-1)}(P_1 \ldots P_m) \rangle$ and $\Delta_{\uplus}^{(n-1)}(P_1 \ldots P_m) = \Delta_{\uplus}^{(n-1)}(P_1) \ldots \Delta_{\uplus}^{(n-1)}(P_m)$. Hence,

$$\langle S_1 \uplus \ldots \uplus S_n \mid P_1 \ldots P_m \rangle = \langle \bigotimes_{i=1}^{n} S_i \mid \prod_{i=1}^{m} \sum_{p+q=n-1} 1^{\otimes p} \otimes P_i \otimes 1^{\otimes q} \rangle.$$

1. For $n > m$, by expanding $\Delta_{\uplus}^{(n-1)}(P_1) \ldots \Delta_{\uplus}^{(n-1)}(P_m)$, one obtains a sum of tensors contening at least one factor equal to 1. For $j = 1, .., n$, the formal power series S_j is proper and the result follows immediatly.
2. For $n = m$, since

$$\prod_{i=1}^{n} \Delta_{\uplus}^{(n-1)}(P_i) = \sum_{\sigma \in \mathfrak{S}_n} \bigotimes_{i=1}^{n} P_{\sigma(i)} + Q,$$

where Q is sum of tensors contening at least one factor equal to 1 and the S_j,'s are proper then $\langle S_1 \otimes \ldots \otimes S_n \mid Q \rangle = 0$. Thus, the result follows.

3. For $n < m$, en tenant compte que, for $j = 1, .., n$, formal power series S_j is proper, the expected follows by expanding the product

$$\prod_{i=1}^{m} \Delta_{\uplus}^{(n-1)}(P_i) = \prod_{i=1}^{m} \sum_{p+q=n-1} 1^{\otimes p} \otimes P_i \otimes 1^{\otimes q}.$$

□

Proposition 18. *1. We have*

$$\log\Big(\sum_{w\in Y^*} w\otimes w\Big) = \sum_{w\in Y^+} w\otimes \pi_1(w) = \sum_{w\in Y^+} \pi_1^*(w)\otimes w,$$

where π_1^* is the adjoint of π_1 and they are given by

$$\begin{aligned}
\pi_1(w) &= \sum_{k\ge 1}\frac{(-1)^{k-1}}{k}\sum_{u_1,\ldots,u_k\in Y^+}\langle w \mid u_1 \uplus \ldots \uplus u_k\rangle u_1\ldots u_k,\\
\pi_1^*(w) &= \sum_{k\ge 1}\frac{(-1)^{k-1}}{k}\sum_{u_1,\ldots,u_k\in Y^+}\langle w \mid u_1\ldots u_k\rangle u_1 \uplus \ldots \uplus u_k.
\end{aligned}$$

In particular, for any $y_k\in Y$, one has

$$\pi_1(y_k) = y_k + \sum_{l\ge 2}\frac{(-1)^{l-1}}{l}\sum_{\substack{j_1,\ldots,j_l\ge 1\\ j_1+\ldots+j_l=k}} y_{j_1}\ldots y_{j_l} \quad \textit{and} \quad \pi_1^*(y_k) = y_k.$$

2. For any $w\in Y^*$, we have

$$\begin{aligned}
w &= \sum_{k\ge 0}\frac{1}{k!}\sum_{u_1,\ldots,u_k\in Y^*}\langle w \mid u_1 \uplus \ldots \uplus u_k\rangle \pi_1(u_1)\ldots\pi_1(u_k),\\
&= \sum_{k\ge 0}\frac{1}{k!}\sum_{u_1,\ldots,u_k\in Y^*}\langle w \mid u_1\ldots u_k\rangle \pi_1^*(u_1) \uplus \ldots \uplus \pi_1^*(u_k).
\end{aligned}$$

Proof. 1. Expanding the logarithm, we have

$$\begin{aligned}
\log\Big(\sum_{w\in Y^*} w\otimes w\Big) &= \sum_{k\ge 1}\frac{(-1)^{k-1}}{k}\Big(\sum_{w\in Y^+} w\otimes w\Big)^k\\
&= \sum_{k\ge 1}\frac{(-1)^{k-1}}{k}\sum_{u_1,\ldots,u_k\in Y^+}(u_1 \uplus \ldots \uplus u_k)\otimes u_1\ldots u_k\\
&= \sum_{w\in Y^+} w\otimes \Big(\sum_{k\ge 1}\frac{(-1)^{k-1}}{k}\\
&\qquad \sum_{u_1,\ldots,u_k\in Y^+}\langle w \mid u_1 \uplus \ldots \uplus u_k\rangle u_1\ldots u_k\Big).
\end{aligned}$$

In the same way,

$$\begin{aligned}
\log\Big(\sum_{w\in Y^*} w\otimes w\Big) &= \sum_{w\in Y^+}\Big(\sum_{k\ge 1}\frac{(-1)^{k-1}}{k}\\
&\qquad \sum_{u_1,\ldots,u_k\in Y^+}\langle w \mid u_1\ldots u_k\rangle u_1 \uplus \ldots \uplus u_k\Big)\otimes w.
\end{aligned}$$

Thus, the expressions of $\pi_1(w)$ and $\pi_1^*(w)$ follow immediatly.

2. Since exp and log are mutually inverse then, by the previous results, one has

$$\begin{aligned}
\sum_{w\in Y^*} w\otimes w &= \sum_{k\geq 0}\frac{1}{k!}\Big(\sum_{w\in Y^+} w\otimes \pi_1(w)\Big)^k\\
&= \sum_{k\geq 0}\frac{1}{k!}\sum_{u_1,\ldots,u_k\in Y^+}(u_1 \uplus \ldots \uplus u_k)\otimes(\pi_1(u_1)\ldots\pi_1(u_k))\\
&= \sum_{w\in Y^+} w\otimes\Big(\sum_{k\geq 1}\frac{1}{k!}\\
&\qquad \sum_{u_1,\ldots,u_k\in Y^+}\langle w\mid u_1\uplus\ldots\uplus u_k\rangle\pi_1(u_1)\ldots\pi_1(u_k)\Big).
\end{aligned}$$

In the same way,

$$\begin{aligned}
\sum_{w\in Y^*} w\otimes w &= \sum_{k\geq 0}\frac{1}{k!}\sum_{u_1,\ldots,u_k\in Y^+}(\pi_1^*(u_1)\uplus\ldots\uplus\pi_1^*(u_k))\otimes(u_1\ldots u_k)\\
&= \sum_{w\in Y^+}\Big(\sum_{k\geq 0}\frac{1}{k!}\\
&\qquad \sum_{u_1,\ldots,u_k\in Y^+}\langle w\mid u_1\ldots u_k\rangle\pi_1^*(u_1)\uplus\ldots\uplus\pi_1^*(u_k)\Big)\otimes w.
\end{aligned}$$

It follows then the expected result. □

5.2 Pair of bases in duality

Definition 16. *Let $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ be the family of $\mathcal{L}ie_{\mathbb{Q}}\langle Y\rangle$ obtained as follows*

$$\begin{aligned}
\Sigma_{y_k} &= \pi_1(y_k) \quad \text{for} \quad k\geq 1,\\
\Sigma_l &= [\Sigma_s,\Sigma_r] \quad \text{for} \quad l\in\mathcal{L}ynX,\ \text{standard factorization of } l=(s,r),
\end{aligned}$$

and the family $\{\Sigma_w\}_{w\in Y^*}$ of $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle Y\rangle)$ (viewed as a $\mathbb{Q}$-module) obtained as follows

$$\begin{aligned}
\Sigma_l &= 1 \quad \text{for} \quad l=1_{Y^*},\\
\Sigma_w &= \Sigma_{l_1}^{i_1}\ldots\Sigma_{l_k}^{i_k} \quad \text{for} \quad w=l_1^{i_1}\ldots l_k^{i_k}, l_1>\ldots>l_k, l_1\ldots,l_k\in\mathcal{L}ynY.
\end{aligned}$$

Let $\{\check{\Sigma}_w\}_{w\in Y^*}$ be the family of the quasi-shuffle algebra (viewed as a $\mathbb{Q}$-module) obtained by duality with $\{\Sigma_w\}_{w\in Y^*}$:

$$\forall u,v\in Y^*,\qquad \langle\check{\Sigma}_v\mid\Sigma_u\rangle=\delta_{u,v}.$$

Proposition 19. *1. For $l\in\mathcal{L}ynY$, the polynomial Σ_l is upper triangular :*

$$\Sigma_l = l+\sum_{v>w,(v)=(l)} c_v v.$$

2. *The families $\{\Sigma_w\}_{w\in Y^*}$ and $\{\check{\Sigma}_w\}_{w\in Y^*}$ are upper and lower triangular respectively. On other words, for any $w \in Y^+$, one has*

$$\Sigma_w = w + \sum_{v>w,(v)=(w)} c_v v \quad \text{and} \quad \check{\Sigma}_w = w + \sum_{v<w,(v)=(w)} d_v v,$$

where, for any $y_k \in Y$ and $w \in Y^*$, (w) denotes the degree of w and $(y_k) = \deg(y_k) = k$.

Proof. 1. Let us prove it by induction on the length of l :

- The result is immediat for $l \in Y$.
- The result is suppose verified for any $l \in \mathcal{L}ynY \cap Y^k$ and $0 \le k \le N$.
- At $N+1$, by the standard factorization (l_1, l_2) of l, one has, by definition, $\Sigma_l = [\Sigma_{l_1}, \Sigma_{l_2}]$ and $l_2 l_1 > l_1 l_2 = l$. By induction hypothesis,

$$\Sigma_{l_1} = l_1 + \sum_{\substack{v>l_1\\(v)=(l_1)}} c_v v \quad \text{and} \quad \Sigma_{l_2} = l_2 + \sum_{\substack{u>l_2\\(v)=(l_2)}} d_u u,$$

$$\Rightarrow \quad \Sigma_l = l + \sum_{\substack{w>l\\(w)=(l)}} e_w w,$$

getting e_w's from c_v's and d_u's. Actually, the Lie bracket gives

$$\begin{aligned}
\Sigma_l &= [l_1, l_2] \\
&+ \sum_{\substack{u>l_2\\(v)=(l_2)}} d_u l_1 u + \sum_{\substack{v>l_1,u>l_2\\(v)=(l_1),(u)=(l_2)}} c_v d_u vu \\
&- \sum_{\substack{v>l_1\\(v)=(l_1)}} c_v l_2 v - \sum_{\substack{v>l_1,u>l_2\\(v)=(l_2),(u)=(l_1)}} c_v d_u uv \\
&= [l_1, l_2] \\
&+ \sum_{\substack{u>l_1l_2\\(v)=(l_1l_2)}} d'_u u + \sum_{\substack{vu>l_1l_2\\(vu)=(l_1l_2)}} c_v d_u vu \\
&- \sum_{\substack{v>l_2l_1\\(v)=(l_2l_1)}} c'_v v - \sum_{\substack{uv>l_2l_1\\(uv)=(l_2l_1)}} c_v d_u uv \\
&= [l_1, l_2] \\
&+ \sum_{\substack{u>l\\(v)=(l)}} d'_u u + \sum_{\substack{vu>l\\(vu)=(l)}} c_v d_u vu \\
&- \sum_{\substack{v>l_2l_1>l\\(v)=(l)}} c'_v v - \sum_{\substack{uv>l_2l_1>l\\(uv)=(l)}} c_v d_u uv.
\end{aligned}$$

Hence, the conclusion follows.

2. Let $w = l_1 \ldots l_k$, with $l_1 > \ldots > l_k$ and $l_1, \ldots, l_k \in \mathcal{L}ynY$. By (ii),

$$\Sigma_{l_i} = l_i + \sum_{\substack{v>l_i\\(v)=(l_i)}} c_{i,v} v \quad \text{and} \quad \Sigma_w = l_1 \ldots l_k + \sum_{\substack{u>w\\(v)=(w)}} d_u u,$$

where the d_u's are obtained from the $c_{i,v}$'s. Hence, the family $\{\Sigma_w\}_{w\in Y^*}$ is upper triangular and, by duality, the family $\{\check{\Sigma}_w\}_{w\in Y^*}$ is lower triangular. $\square$

Theorem 20. *1. The family $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ forms a basis of the free Lie algebra.*

2. The family $\{\Sigma_w\}_{w\in Y^*}$ forms a basis of the free associative algebra $\mathbb{Q}\langle Y\rangle$.

3. The family $\{\check{\Sigma}_w\}_{w\in Y^*}$ generate freely the quasi-shuffle algebra.

4. The family $\{\check{\Sigma}_l\}_{l\in\mathcal{L}ynY}$ forms a transcendence basis of the quasi-shuffle algebra.

Proof. The family $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ of upper triangular polynomials is free and then, by a theorem of Viennot, we get the first result. The second is a direct consequence of the Poincaré-Birkhoff-Witt theorem. By the Cartier-Quillen-Milnor-Moore theorem, we get the third one and the last one is also obtained as consequence of the constructions of the families $\{\check{\Sigma}_l\}_{l\in\mathcal{L}ynY}$ and $\{\check{\Sigma}_w\}_{w\in Y^*}$ of lower triangular polynomials. $\square$

Now, let us clarify the basis $\{\check{\Sigma}_w\}_{w\in Y^*}$ and then the transcendence basis $\{\check{\Sigma}_l\}_{l\in\mathcal{L}ynY}$ of the quasi-shuffle algebra $(\mathbb{Q}\langle Y\rangle, \uplus)$ as follows

Theorem 21. *We have*

1. *For $w = 1_{Y^*}$, $\check{\Sigma}_w = 1$.*
2. *For any $w = l_1^{i_1}\ldots l_k^{i_k}$, with $l_1,\ldots,l_k \in \mathcal{L}ynY$ and $l_1 > \ldots > l_k$,*
$$\check{\Sigma}_w = \frac{\check{\Sigma}_{l_1}^{\uplus\, i_1} \uplus \ldots \uplus \check{\Sigma}_{l_k}^{\uplus\, i_k}}{i_1!\ldots i_k!}.$$
3. *For any $y\in Y, \check{\Sigma}_y = \pi_1^*(y)$.*
4. *For any $l = yu \in \mathcal{L}ynY - Y$, $\check{\Sigma}_l = y\check{\Sigma}_u$.*

Proof. 1. Since $\Sigma_{1_{Y^*}} = 1$ then $\check{\Sigma}_{1_{Y^*}} = 1$.

2. Let $u = u_1\ldots u_n = l_1^{i_1}\ldots l_k^{i_k}, v = v_1\ldots v_m = h_1^{j_1}\ldots h_p^{j_p}$ with $l_1\ldots,l_k$, $h_1,\ldots,h_p,u_1,\ldots,u_n,v_1,\ldots,v_m \in \mathcal{L}ynY, l_1 > \ldots > l_k, h_1 > \ldots > h_p$, $u_1 \ge \ldots \ge u_n, v_1 \ge \ldots \ge v_m$ and $i_1+\ldots+i_k = n, j_1+\ldots+j_p = m$. Hence, if $m\ge 2$ (resp. $n\ge 2$) then $v\notin \mathcal{L}ynY$ (resp. $u\notin\mathcal{L}ynY$).

   Since
$$\langle \check{\Sigma}_{u_1} \uplus \ldots \uplus \check{\Sigma}_{u_n} \mid \prod_{i=1}^n \Sigma_{u_i}\rangle = \langle \check{\Sigma}_{u_1}\otimes\ldots\otimes\check{\Sigma}_{u_n} \mid \Delta_{\uplus}^{(n-1)}(\Sigma_{v_1}\ldots\Sigma_{v_m})\rangle$$
   then many cases occur :

(a) Case $n > m$. By Lemma 23(1), $\langle \check{\Sigma}_{u_1} ⧢ \ldots ⧢ \check{\Sigma}_{u_n} \mid \Sigma_{v_1} \ldots \Sigma_{v_m} \rangle = 0$.

(b) Case $n = m$. By Lemma 23(2), one has

$$\begin{aligned}\langle \check{\Sigma}_{u_1} ⧢ \ldots ⧢ \check{\Sigma}_{u_n} \mid \prod_{i=1}^{n} \Sigma_{v_i} \rangle &= \sum_{\sigma \in \check{\Sigma}_n} \prod_{i=1}^{n} \langle \check{\Sigma}_{u_i} \mid \Sigma_{v_{\sigma(i)}} \rangle \\ &= \sum_{\sigma \in \check{\Sigma}_n} \prod_{i=1}^{n} \delta_{\check{\Sigma}_{u_i}, \Sigma_{v_{\sigma(i)}}}.\end{aligned}$$

Thus, if $u \neq v$ then $(u_1, \ldots, u_n) \neq (v_1, \ldots, v_n)$ then the second member is vanishing else, *i.e.* $u = v$, the second member equals 1 because the factorization by Lyndon words is unique.

(c) Case $n < m$. By Lemma 23(3), let us consider the following language over the alphabet $\mathcal{A} = \{\Sigma_{v_1}, \ldots, \Sigma_{v_m}\}$:

$$\mathcal{M} = \{w \in \mathcal{A}^* | w = \Sigma_{v_{j_1}} \ldots \Sigma_{v_{j_{|w|}}}, j_1 < \ldots < j_{|w|}, |w| \geq 1\},$$

and the morphism $\mu : \mathbb{Q}\langle \mathcal{A} \rangle \longrightarrow \mathbb{Q}\langle Y \rangle$. We get :

$$\begin{aligned}\langle ⧢_{i=1}^{n} \check{\Sigma}_{u_i} \mid \prod_{i=1}^{n} \Sigma_{u_i} \rangle &= \sum_{\substack{w_1, \ldots, w_m \in \mathcal{M} \\ |w_1| + \ldots + |w_n| = m \\ \forall i,j=1,..m, \mathrm{alp}(w_i) \cap \mathrm{alp}(w_j) = \emptyset}} \prod_{i=1}^{n} \langle \check{\Sigma}_{u_i} \mid \mu(w_i) \rangle \\ &= 0.\end{aligned}$$

Because in this product, on one hand, there exists at least one $w_i \in \mathcal{M}$, $|w_i| \geq 2$, corresponding to $\Sigma_{v_{j_1}} \ldots \Sigma_{v_{j_{|w_i|}}} = \mu(w_i)$ such that $v_{j_1} \geq \ldots \geq v_{j_{|w_i|}}$ and on other hand, $\nu_i := v_{j_1} \ldots v_{j_{|w_i|}} \notin \mathcal{L}ynY$ and $u_i \in \mathcal{L}ynY$.

By consequent,

$$\langle \check{\Sigma}_u \mid \Sigma_v \rangle = \langle \frac{\check{\Sigma}_{l_1}^{⧢ i_1} ⧢ \ldots ⧢ \check{\Sigma}_{l_k}^{⧢ i_k}}{i_1! \ldots i_k!} \mid \Sigma_{h_1}^{j_1} \ldots \Sigma_{h_p}^{j_p} \rangle = \delta_{u,v}.$$

3. For any $l \in Y, \Sigma_l = \pi_1(l), \check{\Sigma}_l = \pi_1^*(l)$ and π_1, π_1^* are mutually adjoint.

4. By decomposing bu in the PBW-Lyndon basis, we get on one hand,

$$\begin{aligned}bu &= \sum_{w \in Y^*} \langle \check{\Sigma}_w \mid bu \rangle \Sigma_w + \sum_{aw \in \mathcal{L}ynY} \langle \check{\Sigma}_{aw} \mid bu \rangle \Sigma_{aw} \\ &+ \text{ sum of decreasinge products of length} \geq 2, \text{ of Lie polynomials.}\end{aligned}$$

Since for any $aw = l_1^{i_1} \ldots l_k^{i_k} \notin \mathcal{L}ynY$ $(k \geq 2)$ with $l_1, \ldots, l_k \in \mathcal{L}ynY$ and $l_1 > \ldots > l_k$, then $\Sigma_{aw} = \Sigma_{l_1} \ldots \Sigma_{l_k}$. Since (see Proposition 19)

$$\Sigma_w = w + \sum_{v > w, (v) = (w)} c_v v \quad \text{and} \quad \Sigma_{bw} = bw + \sum_{v > bw, (v) = (bw)} d_v v$$

then, by decomposing u in the PBW-Lyndon basis and then by multiplying by b, we get on other hand,

$$\begin{aligned}
bu &= \sum_{w\in Y^*} \langle \check{\Sigma}_w \mid u\rangle b\Sigma_w \\
&= \sum_{w\in Y^*} \langle \check{\Sigma}_w \mid u\rangle \Bigg(\Sigma_{bw} - \sum_{v>bw,(v)=(bw)} d_v v\Bigg) \\
&= \sum_{w\in Y^*} \langle \check{\Sigma}_w \mid u\rangle \Sigma_{bw} - \sum_{w\in Y^*} \sum_{v>bw,(v)=(bw)} d_v \sum_{w'\in Y^*} \langle \check{\Sigma}_{w'} \mid v\rangle \Sigma_{w'} \\
&\quad \text{(by decomposing } v \text{ in PBW-Lyndon basis)} \\
&= \sum_{w\in Y^*} \langle \check{\Sigma}_w \mid u\rangle \Sigma_{bw} - \sum_{w\in Y^*} \sum_{w'\in Y^*} \sum_{v>bw,(v)=(bw)} d_v \langle \check{\Sigma}_{w'} \mid v\rangle \Sigma_{w'} \\
&= \sum_{bw\in \mathcal{L}ynY} \langle \check{\Sigma}_w \mid u\rangle \Sigma_{bw} - \sum_{w'\in \mathcal{L}ynY} \sum_{w\in Y^*} \sum_{v>bw,(v)=(bw)} d_v \langle \check{\Sigma}_{w'} \mid v\rangle \Sigma_{w'} \\
&+ \quad \text{sum of decreasinge products, of length} \geq 2, \text{ of Lie polynomials.}
\end{aligned}$$

After splitting these two sums on two disjoint supports, one has

- for any $bw = l_1^{i_1} \ldots l_n^{i_n} \notin \mathcal{L}ynY$ $(n \geq 2)$ with $l_1, \ldots, l_n \in \mathcal{L}ynY$ verifying $l_1 > \ldots > l_n$ and $\Sigma_{aw} = \Sigma_{l_1}^{i_n} \ldots \Sigma_{l_n}^{i_n}$.
- for any $w' = \lambda_1^{j_1} \ldots \lambda_m^{j_m} \notin \mathcal{L}ynY$ $(m \geq 2)$ with $\lambda_1, \ldots, \lambda_m \in \mathcal{L}ynY$ verifying $\lambda_1 > \ldots > \lambda_m$ and $\Sigma_{w'} = \Sigma_{\lambda_1}^{j_1} \ldots \Sigma_{\lambda_m}^{j_m}$.

In the second sum, since each word v is great than the Lyndon word bw then the Lie polynomial Σ_{bw} does not appear in the decomposition, in the PBW-Lyndon basis, of v. More precisely, (see Proposition 19)

$$\begin{aligned}
\check{\Sigma}_{w'} &= w' + \sum_{w'>v',(w')=(v')} e_{v'} v' \quad \text{with } e_{v'} \geq 0, \\
\Rightarrow \quad \langle \check{\Sigma}_{w'} \mid v\rangle &= \langle w' \mid v\rangle + \sum_{w'>v',(w')=(v')} e_{v'} \langle v' \mid v\rangle.
\end{aligned}$$

In particular (for $w' = bw \in \mathcal{L}ynY$), the coefficient of the Lie polynomial Σ_{bw} in the decomposition of v $(> bw)$ is vanishing :

$$\langle \check{\Sigma}_{bw} \mid v\rangle = \langle bw \mid v\rangle + \sum_{v>bw>v',(bw)=(v')} e_{v'} \langle v' \mid v\rangle = 0.$$

Thus, by identifying the coefficients in these two expressions of Lyndon word bu, one has $\langle \check{\Sigma}_{aw} \mid bu\rangle = \delta_{a,b} \langle \check{\Sigma}_w \mid u\rangle$. In other words, $\check{\Sigma}_{bw} = b\check{\Sigma}_w$. □

Corollary 16. *1. For $w \in Y^+$, the polynomial $\check{\Sigma}_w$ is proper and homogenous of degree $|w|$, for $\deg(y_i) = i$, and of rational positive coefficients.*

2.

$$\sum_{w\in Y^*} w\otimes w = \sum_{w\in Y^*} \check{\Sigma}_w\otimes\Sigma_w = \prod_{l\in\mathcal{L}ynY}^{\searrow} \exp(\check{\Sigma}_l\otimes\Sigma_l).$$

3. The family $\mathcal{L}ynY$ forms a transcendence basis[30] *of the quasi-shuffle algebra and the family of proper polynomials of rational positive coefficients defined by, for any $w = l_1^{i_1}\ldots l_k^{i_k}$ with $l_1 > \ldots > l_k$ and $l_1,\ldots,l_k\in\mathcal{L}ynY$,*

$$\chi_w = \frac{l_1^{\uplus i_1}\uplus\ldots\uplus l_k^{\uplus i_k}}{i_1!\ldots i_k!}$$

forms a basis of the quasi-shuffle algebra.

4. Let $\{\xi_w\}_{w\in Y^*}$ be the basis of the envelopping algebra $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X\rangle)$ obtained by duality with the basis $\{\chi_w\}_{w\in Y^*}$:

$$\forall u,v\in Y^*,\qquad \langle\chi_v\mid\xi_u\rangle = \delta_{u,v}.$$

Then the family $\{\xi_l\}_{l\in\mathcal{L}ynY}$ forms a basis of the free Lie algebra $\mathcal{L}ie_{\mathbb{Q}}\langle Y\rangle$.

Proof. 1. The proof can be done by induction on the length of w using the fact that the product $\uplus$ conserve the property, l'homogenity and rational positivity of the coefficients.

2. Expressing w in the basis $\{\check{\Sigma}_w\}_{w\in Y^*}$ of the quasi-shuffle algebra and then in the basis $\{\Sigma_w\}_{w\in Y^*}$ of the envelopping algebra, we obtain successively

$$\begin{aligned}
\sum_{w\in Y^*} w\otimes w &= \sum_{w\in Y^*}\left(\sum_{u\in X^*}\langle\Sigma_u\mid w\rangle\check{\Sigma}_u\right)\otimes w\\
&= \sum_{u\in Y^*}\check{\Sigma}_u\otimes\left(\sum_{w\in X^*}\langle\Sigma_u\mid w\rangle w\right)\\
&= \sum_{u\in Y^*}\check{\Sigma}_u\otimes\Sigma_u\\
&= \sum_{\substack{l_1>\ldots>l_k\\ i_1,\ldots,i_k\geq 1}} \frac{\check{\Sigma}_{l_1}^{\sqcup\!\sqcup\, i_1}\uplus\ldots\uplus\check{\Sigma}_{l_k}^{\sqcup\!\sqcup\, i_k}}{i_1!\ldots i_k!}\otimes\Sigma_{l_1}^{i_1}\ldots\Sigma_{l_k}^{i_k}\\
&= \prod_{l\in\mathcal{L}ynY}^{\searrow}\sum_{i\geq 0}\frac{\check{\Sigma}_l^{\uplus i}}{i!}\otimes\Sigma_l^i\\
&= \prod_{l\in\mathcal{L}ynY}^{\searrow}\exp(\check{\Sigma}_l\otimes\Sigma_l).
\end{aligned}$$

[30]This result is an analogous of a Radford theorem (see [45]). Thus the bases $\mathcal{L}ynY$ and $\{\check{\Sigma}_l\}_{l\in\mathcal{L}ynY}$ belong to the class of Radford bases, *i.e.* the class of trancensdence bases, of the quasi-shuffle algebra, as well as the bases $\mathcal{L}ynX$ and $\{S_l\}_{l\in\mathcal{L}ynX}$ belong to the class of Radford bases of the shuffle algebra.

3. For $w = l_1^{i_1} \ldots l_k^{i_k}$ with $l_1, \ldots, l_k \in \mathcal{L}ynY$ and $l_1 > \ldots > l_k$, by Proposition 19, the polynomial of rational positive coefficients $\check{\Sigma}_w$ is lower triangular :
$$\check{\Sigma}_w = \frac{\check{\Sigma}_{l_1}^{\text{⧢} i_1} \text{⧢} \ldots \text{⧢} \check{\Sigma}_{l_k}^{\text{⧢} i_k}}{i_1! \ldots i_k!} = w + \sum_{v<w,(v)=(w)} c_v v.$$
In particular, for any $l_j \in \mathcal{L}ynY$, $\check{\Sigma}_{l_j}$ is lower triangular :
$$\check{\Sigma}_{l_j} \quad = \quad l_j + \sum_{v<l_j,(v)=(l_j)} c_v v.$$
Hence, $\check{\Sigma}_w = \chi_w + \chi'_w$, where χ'_w is a proper polynomial of $\mathbb{Q}\langle Y\rangle$ of rational positive coefficients. We deduce then the support of χ_w contains words which are less than w and $\langle \chi_w \mid w\rangle = 1$. Thus, the proper polynomial χ_w of rational positive coefficients is lower triangular :
$$\begin{aligned} \chi_w \quad &= \quad w + \sum_{v<w,(v)=(w)} c_v v, \\ \Rightarrow \quad \forall l \in \mathcal{L}ynY, \quad \chi_l \quad &= \quad l + \sum_{v<l,(v)=(l)} c_v v. \end{aligned}$$
It follows then expected results.

4. By duality, for $w \in Y^*$, the proper polynomial ξ_w is upper triangular. In particular, for any $l \in \mathcal{L}ynY$, the proper polynomial ξ_l is upper triangular :
$$\xi_l \quad = \quad l + \sum_{v>l,(v)=(l)} d_v v.$$
Hence, the family $\{\xi_l\}_{l\in\mathcal{L}ynY}$ is free and its elements verify an analogous of the generalized criterion of Friedrichs :
   - for $w \in \mathcal{L}ynY$, one has $\langle \chi_w \mid \xi_l\rangle = \delta_{w,l}$,
   - for $w \notin \mathcal{L}ynY$, $w = l_1 \ldots l_n$ with $l_1, \ldots, l_n \in \mathcal{L}ynY$ and $l_1 > \ldots > l_n$, one has $\langle \chi_w \mid \xi_l\rangle = \langle \chi_{l_1} \text{⧢} \ldots \text{⧢} \chi_{l_n} \mid \xi_l\rangle = 0$.

   Moreover, the polynomials ξ_l's are primitive : by Corollary 16(3), one has
$$\begin{aligned} \Delta_{\text{⧢}}(\xi_l) \quad &= \quad \sum_{u,v\in Y^*} \langle u \text{⧢} v \mid \xi_l\rangle u \otimes v \\ &= \quad \sum_{u\in Y^+} \langle u \text{⧢} 1_{Y^*} \mid \xi_l\rangle u \otimes 1_{Y^*} + \sum_{v\in Y^+} \langle 1_{Y^*} \text{⧢} v \mid \xi_l\rangle 1_{Y^*} \otimes v \\ &+ \quad \sum_{u,v\in Y^+} \langle u \text{⧢} v \mid \xi_l\rangle u \otimes v + \langle 1_{Y^*} \text{⧢} 1_{Y^*} \mid \xi_l\rangle 1_{Y^*} \otimes 1_{Y^*} \\ &= \quad \xi_l \otimes 1_{Y^*} + 1_{Y^*} \otimes \xi_l. \end{aligned}$$
Because, after decomposing u and v on the basis $\{\chi_l\}_{l\in\mathcal{L}ynY}$ and by the previous criterion, the third term is vanishing. The last one is also vanishing since the ξ_l's are proper. By a theorem of Viennot, we obtain then the expected result.

□

5.3 Proof of Theorem 2

Applying the tensor product of isomorphisms $\mathrm{H} \otimes \mathrm{Id}$ (Proposition 1) on the diagonal series (Corollary 16(ii)), the infinite factorization, by Lyndon words, of the noncommutative generating series of harmonic sums follows[31] :

$$\mathrm{H}(N) = \sum_{w \in Y^*} \mathrm{H}_w(N)\, w = \prod_{l \in \mathcal{L}ynY}^{\searrow} \exp(\mathrm{H}_{\check{\Sigma}_l}(N)\, \Sigma_l). \tag{111}$$

6 Annexe B : differential realization

To facilitate reading, the following results are placed in this Annex which can be skipped by readers already familiar with the techniques developed by Fliess (and adapted by us for studies in this paper).

6.1 Polysystem and convergence criterion

6.1.1 Serial estimates from above

Here, generalizing a little, $\mathbb{K}$ is supposed a $\mathbb{C}$-algebra and a complete normed vector space equipped with a norm denoted by $\|.\|$.

For any $n \in \mathbb{N}, X^{\geq n}$ denotes the set of words over X of length greater than or equal to n. The set of formal power series (resp. polynomials) on X, is denoted by $\mathbb{K}\langle\langle X\rangle\rangle$ (resp. $\mathbb{K}\langle X\rangle$).

Definition 17 ([25, 39]). *Let ξ, χ be real positive functions over X^*. Let $S \in \mathbb{K}\langle\langle X\rangle\rangle$.*

1. *S will be said $\xi-$*exponentially bounded from above *if it verifies*

$$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \qquad \|\langle S \mid w\rangle\| \leq K\xi(w)/|w|!.$$

We denote by $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$ the set of formal power series in $\mathbb{K}\langle\langle X\rangle\rangle$ which are $\xi-$exponentially bounded from above.

2. *S verifies the $\chi-$*growth condition *if it satisfies*

$$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \qquad \|\langle S \mid w\rangle\| \leq K\chi(w)\,|w|!.$$

We denote by $\mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$ the set of formal power series in $\mathbb{K}\langle\langle X\rangle\rangle$ verifying the $\chi-$growth condition.

[31] This proof omitted in previous versions uses mainly the results presented in this annexe that have not been published earlier but have already been presented at various workshops. It is an analogous way to obtain the infinite factorization, by Lyndon words over the alphabet X, of the noncommutative generating series of polylogarithms (see Theorem 3) by applying the tensor product of isomorphisms $\mathrm{Li} \otimes \mathrm{Id}$ (see Proposition 1) on the diagonal series, over X.

Lemma 24. *We have*

$$R = \sum_{w\in X^*} |w|!\, w \quad\Rightarrow\quad \langle R^{⧢ 2} \mid w\rangle = \sum_{\substack{u,v\in X^*\\ \mathrm{supp}(u ⧢ v)\ni w}} |u|!\,|v|! \le 2^{|w|}\,|w|!.$$

Proof. One has

$$\begin{aligned}\sum_{\substack{u,v\in X^*\\ \mathrm{supp}(u ⧢ v)\ni w}} |u|!\,|v|! &= \sum_{k=0}^{|w|} \sum_{\substack{|u|=k,|v|=|w|-k\\ \mathrm{supp}(u ⧢ v)\ni w}} k!(|w|-k)!\\ &= \sum_{k=0}^{|w|} \binom{|w|}{k} k!(|w|-k)!\\ &= \sum_{k=0}^{|w|} |w|!\\ &= (1+|w|)\,|w|!.\end{aligned}$$

By induction on the length of w, one has $1+|w| \le 2^{|w|}$. It follows the expected result. □

Proposition 20. *Let S_1 and S_2 verifying the growth condition. Then S_1+S_2 and $S_1 ⧢ S_2$ also verifies the growth condition.*

Proof. The proof for S_1+S_2 is immediate.

Next, since $\|\langle S_i \mid w\rangle\| \le K_i\chi_i(w)\,|w|!$, for $i=1$ or 2 and for $w\in X^*$, then[32]

$$\begin{aligned}\langle S_1 ⧢ S_2 \mid w\rangle &= \sum_{\mathrm{supp}(u ⧢ v)\ni w} \langle S_1 \mid u\rangle\langle S_2 \mid v\rangle,\\ \Rightarrow\quad \|\langle S_1 ⧢ S_2 \mid w\rangle\| &\le K_1K_2 \sum_{\substack{u,v\in X^*\\ \mathrm{supp}(u ⧢ v)\ni w}} (\chi_1(u)\,|u|!)(\chi_2(v)\,|v|!).\end{aligned}$$

Let $K = K_1K_2$ and let χ be a real positive function over X^* such that

$$\forall w\in X^*,\qquad \chi(w) = \max\{\chi_1(u)\chi_2(v) \mid u,v\in X^* \text{ and } \mathrm{supp}(u ⧢ v)\ni w\}.$$

With the notations in Lemma 24, we get

$$\|\langle S_1 ⧢ S_2 \mid w\rangle\| \le K\chi(w)\langle R^{⧢ 2} \mid w\rangle.$$

Hence, $S_1 ⧢ S_2$ verifies the χ'-growth condition with χ' defined as $\chi'(w) = 2^{|w|}\chi(w)$. □

Definition 18 ([25, 39])**.** *Let ξ be a real positive function defined over X^*, S will be said* ξ-exponentially continuous *if it is continuous over $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$. The set of formal power series which are ξ-exponentially continuous is denoted by $\mathbb{K}^{\xi-ec}\langle\langle X\rangle\rangle$.*

[32] $\langle S_1 ⧢ S_2 \mid w\rangle$ is the coefficient of the word w in the power series $S_1 ⧢ S_2$.

Lemma 25 ([25, 39])**.** *For any real positive function ξ defined over X^*, we have $\mathbb{K}\langle X\rangle \subset \mathbb{K}^{\xi-ec}\langle\langle X\rangle\rangle$. Otherwise, for $\xi = 0$, we get $\mathbb{K}\langle X\rangle = \mathbb{K}^{0-ec}\langle\langle X\rangle\rangle$. Hence, any polynomial is $0-$exponentially continuous.*

Proposition 21 ([25, 39])**.** *Let ξ, χ be a real positive functions over X^* and let $P \in \mathbb{K}\langle X\rangle$.*

1. *Let $S \in \mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$. The right residual of S by P belongs to $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$.*
2. *Let $R \in \mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$. The concatenation SR belongs to $\mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$.*

Proof. 1. Since $S \in \mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$ then

$$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \qquad \|\langle S \mid w\rangle\| \leq K\xi(w)/|w|!.$$

If $u \in \mathrm{supp}(P) := \{w \in X^* \mid \langle P \mid w\rangle \neq 0\}$ then, for any $w \in X^*$, one has $\langle S \triangleright u \mid w\rangle = \langle S \mid uw\rangle$ and $S \triangleright u$ belongs to $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$:

$$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \qquad \|\langle S \triangleright u \mid w\rangle\| \leq [K\xi(u)]\xi(w)/|w|!.$$

It follows then $S \triangleright P$ is $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$ by taking $K_1 = K \max_{u\in\mathrm{supp}(P)} \xi(u)$.

2. Since $R \in \mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$ then

$$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \qquad \|\langle S \mid w\rangle\| \leq K\chi(w)\,|w|!.$$

Let $v \in \mathrm{supp}(P)$ such that $v \neq \epsilon$. Since, for any $w \in X^*$, Rv belongs to $\mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$ and one has $\langle Rv \mid w\rangle = \langle R \mid v \triangleleft w\rangle$:

$$\begin{aligned}\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \quad \|\langle R \mid v \triangleleft w\rangle\| &\leq K\chi(v \triangleleft w)(|w| - |v|)!\\ &\leq K\,|w|\,\chi(w)/\chi(v).\end{aligned}$$

Note that if $v\triangleleft w = 0$ then $\langle Rv \mid w\rangle = 0$ and the previous conclusion holds. It follows then RP is $\mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$ by taking $K_2 = K \min_{v\in\mathrm{supp}(P)} \chi(v)^{-1}$.

□

Proposition 22 ([25, 39])**.** *Two real positive morphisms over X^*, ξ and χ are assumed to verify the condition*

$$\sum_{x\in X} \chi(x)\xi(x) \quad < \quad 1.$$

Then for any $F \in \mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$, F is continuous over $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$.

Proof. If ξ, χ verify the upper bound condition then the following power series

$$\sum_{w\in X^*} \chi(w)\xi(w) \quad = \quad \Big(\sum_{x\in X} \chi(x)\xi(x)\Big)^*$$

is well defined. If $F \in \mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$ and $C \in \mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$ then there exists $K_i \in \mathbb{R}_+$ and $n_i \in \mathbb{N}$ such that for any $w \in X^{\geq n_i}, i = 1, 2$, one has

$$\|\langle F \mid w\rangle\| \leq K_1\chi(w)\,|w|! \quad \text{and} \quad \|\langle C \mid w\rangle\| \leq K_2\xi(w)/|w|!.$$

Hence,

$$\begin{aligned} &\forall w \in X^*, |w| \geq \max\{n_1, n_2\}, \quad \|\langle F|w\rangle\langle C|w\rangle\| \leq K_1K_2\chi(w)\xi(w), \\ \Rightarrow \quad &\sum_{w\in X^*} \|\langle F|w\rangle\langle C|w\rangle\| \leq K_1K_2 \sum_{w\in X^*} \chi(w)\xi(w) = K_1K_2\left(\sum_{x\in X} \chi(x)\xi(x)\right)^*. \end{aligned}$$

□

6.1.2 Upper bounds *à la* Cauchy

Let $q_1, \ldots, q_n$ be commutative indeterminates over $\mathbb{C}$. The algebra of formal power series (resp. polynomials) over $\{q_1, \ldots, q_n\}$ with coefficients in $\mathbb{C}$ is denoted by $\mathbb{C}[\![q_1, \ldots, q_n]\!]$ (resp. $\mathbb{C}[q_1, \ldots, q_n]$).

Definition 19 ([25, 39])**.** *Let*

$$f = \sum_{i_1,\ldots,i_n\geq 0} f_{i_1,\ldots,i_n} q_1^{i_1} \ldots q_n^{i_n} \in \mathbb{C}[\![q_1, \ldots, q_n]\!].$$

We set

$$\begin{array}{rcl} E(f) & := & \{\rho \in \mathbb{R}_+^n : \exists C_f \in \mathbb{R}_+ \ s.t.\ \forall i_1, \ldots, i_n \geq 0, |\, f_{i_1,\ldots,i_n} |\, \rho_1^{i_1} \ldots \rho_n^{i_n} \leq C_f\} \\ \breve{E}(f) & : & \textit{interior of } E(f) \textit{ in } \mathbb{R}^n. \\ \mathrm{CV}(f) & := & \{q \in \mathbb{C}^n : (|q_1|, \ldots, |q_n|) \in \breve{E}(f)\} \quad : \quad \textit{convergence domain of } f. \end{array}$$

The power series f is to be said convergent *if* $\mathrm{CV}(f) \neq \emptyset$*. Let $\mathcal{U}$ be an open domain in $\mathbb{C}^n$ and let $q \in \mathbb{C}^n$. The power series f is to be said convergent on q (resp. over $\mathcal{U}$) if $q \in \mathrm{CV}(f)$ (resp. $\mathcal{U} \subset \mathrm{CV}(f)$). We set*

$$\mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!] \quad = \quad \{f \in \mathbb{C}[\![q_1, \ldots, q_n]\!] : \mathrm{CV}(f) \neq \emptyset\}.$$

Let $q \in \mathrm{CV}(f)$. There exists the constants C_f, ρ and $\bar\rho$ such that $|q_1| < \bar\rho < \rho, \ldots, |q_n| < \bar\rho < \rho$ and, for $i_1 \ldots, i_n \geq 0$, $|\, f_{i_1,\ldots,i_n} |\, \rho_1^{i_1} \ldots \rho_n^{i_n} \leq C_f$. The convergence modulus *of f at q is $(C_f, \rho, \bar\rho)$.*

Suppose that $\mathrm{CV}(f) \neq \emptyset$ and let $q \in \mathrm{CV}(f)$. If $(C_f, \rho, \bar\rho)$ is a convergence modulus of f at q then $|\, f_{i_1,\ldots,i_n} q_1^{i_1} \ldots q_n^{i_n} | \leq C_f(\bar\rho_1/\rho_1)^{i_1} \ldots (\bar\rho_1/\rho_1)^{i_n}$. Hence, at q, the power series f is majored termwise by

$$C_f \prod_{k=0}^{m} \left(1 - \frac{\bar\rho_k}{\rho_k}\right)^{-1}. \tag{112}$$

Hence, f is uniformly absolutely convergent in $\{q \in \mathbb{C}^n : \mid q_1 \mid < \bar{\rho}, \ldots, \mid q_n \mid < \bar{\rho}\}$ which is an open domain in $\mathbb{C}^n$. Thus, $\mathrm{CV}(f)$ is an open domain in $\mathbb{C}^n$. Since the partial derivation of order $j_1, \ldots, j_n \geq 0$ of f is estimated by

$$\|D_1^{j_1} \ldots D_n^{j_n} f\| \quad \leq \quad C_f \frac{\partial^{j_1+\ldots+j_n}}{\partial \bar{\rho}^{j_1+\ldots+j_n}} \prod_{k=0}^{m} \left(1 - \frac{\bar{\rho}_k}{\rho_k}\right)^{-1}. \tag{113}$$

Proposition 23 ([25]). *We have* $\mathrm{CV}(f) \subset \mathrm{CV}(D_1^{j_1} \ldots D_n^{j_n} f)$.

Let $f \in \mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!]$ and let $\{A_i\}_{i=0,1}$ be a polysystem defined as follows

$$A_i(q) = \sum_{j=1}^{n} A_i^j(q) \frac{\partial}{\partial q_j}, \quad \text{with} \quad A_i^j(q) \in \mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!], j = 1, \ldots, n. \tag{114}$$

Lemma 26 ([20]). *For $i = 0, 1$ and $j = 1, .., n$, one has $A_i \circ q_j = A_i^j(q)$. Thus,*

$$\forall i = 0, 1, \quad A_i(q) \quad = \quad \sum_{j=1}^{n} (A_i \circ q_j) \frac{\partial}{\partial q_j}.$$

Let $(\rho, \bar{\rho}, C_f), \{(\rho, \bar{\rho}, C_i)\}_{i=0,1}$ be respectively the convergence modulus at

$$q \quad \in \quad \mathrm{CV}(f) \bigcap_{\substack{i=0,1 \\ j=1,..,n}} \mathrm{CV}(A_i^j) \tag{115}$$

of f and $\{A_i^j\}_{j=1,..,n}$. Let us consider the following monoid morphisms

$$\mathcal{A}(\epsilon) = \text{identity} \quad \text{and} \quad C(\epsilon) = 1, \tag{116}$$

$$\forall w = v x_i, x_i \in X, v \in X^*, \quad \mathcal{A}(w) = \mathcal{A}(v) A_i \quad \text{and} \quad C(w) = C(v) C_i. \tag{117}$$

Lemma 27 ([19]). *For any word w, $\mathcal{A}(w)$ is continuous over $\mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!]$ and, for any $f, g \in \mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!]$, one has*

$$\mathcal{A}(w) \circ (fg) \quad = \quad \sum_{u,v \in X^*} \langle u \sqcup\!\sqcup v \mid w \rangle \; (\mathcal{A}(u) \circ f)(\mathcal{A}(v) \circ g).$$

These notations are extended, by linearity, to $\mathbb{K}\langle X \rangle$ and we will denote $\mathcal{A}(w) \circ f_{|q}$ the evaluation of $\mathcal{A}(w) \circ f$ at q.

Definition 20 ([19]). *Let $f \in \mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!]$. The generating series of the polysystem $\{A_i\}_{i=0,1}$ and of the observation f is given by*

$$\sigma f \quad := \quad \sum_{w \in X^*} \mathcal{A}(w) \circ f \; w \quad \in \quad \mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!]\langle\!\langle X \rangle\!\rangle.$$

$$\sigma f_{|_q} \quad := \quad \sum_{w \in X^*} \mathcal{A}(w) \circ f_{|q} \; w \quad \in \quad \mathbb{C}\langle\!\langle X \rangle\!\rangle.$$

The last generating series is called Fliess generating series *of the polysystem $\{A_i\}_{i=0,1}$ and of the observation f at q.*

Lemma 28 ([19])**.** *Let $\{A_i\}_{i=0,1}$ be a polysystem. Then, the map*

$$\sigma : (\mathbb{C}^{\mathrm{cv}}[\![q_1,\ldots,q_n]\!],.) \quad\longrightarrow\quad (\mathbb{C}^{\mathrm{cv}}[\![q_1,\ldots,q_n]\!]\langle\!\langle X\rangle\!\rangle, ⧢),$$

is an algebra morphism, i.e. *for any $f,g \in \mathbb{C}^{\mathrm{cv}}[\![q_1,\ldots,q_n]\!]$ and $\mu,\nu \in \mathbb{C}$, one has $\sigma(\nu f + \mu h) = \nu\sigma f + \mu\sigma g$ and $\sigma(fg) = \sigma f ⧢ \sigma g$.*

Lemma 29 ([20])**.** *Let $\{A_i\}_{i=0,1}$ be a polysystem and $f \in \mathbb{C}^{\mathrm{cv}}[\![q_1,\ldots,q_n]\!]$. Then*

$$\begin{array}{rcl} \forall x_i \in X, \quad \sigma(A_i \circ f) = x_i \triangleleft \sigma f & \in & \mathbb{C}^{\mathrm{cv}}[\![q_1,\ldots,q_n]\!]\langle\!\langle X\rangle\!\rangle \\ \forall w \in X^*, \quad \sigma(\mathcal{A}(w) \circ f) = w \triangleleft \sigma f & \in & \mathbb{C}^{\mathrm{cv}}[\![q_1,\ldots,q_n]\!]\langle\!\langle X\rangle\!\rangle. \end{array}$$

Lemma 30 ([25])**.** *Let $\tau = \min_{1\le k\le n} \rho_k$ and $r = \max_{1\le k\le n} \bar{\rho}_k/\rho_k$. We have*

$$\begin{array}{rcl} \|\mathcal{A}(w) \circ f\| & \le & C_f \dfrac{(n+1)}{(1-r)^n} \dfrac{C(w)\ |w|!}{\binom{n+|w|-1}{|w|}} \left[\dfrac{n}{\tau(1-r)^{n+1}}\right]^{|w|} \\ & \le & C_f \dfrac{(n+1)}{(1-r)^n} C(w) \left[\dfrac{n}{\tau(1-r)^{n+1}}\right]^{|w|} |w|!. \end{array}$$

Theorem 22 ([25])**.** *Let $K = C_f(n+1)(1-r)^{-n}$ and let χ be the real positive function defined over X^* by*

$$\forall i = 0,1, \quad \chi(x_i) \quad = \quad \frac{C_i n}{\tau(1-r)^{(n+1)}}.$$

Then the generating series σf of the polysystem $\{A_i\}_{i=0,1}$ and of the observation f satisfies the $\chi-$growth condition.

It is the same for the Fliess generating series $\sigma f_{|q}$ of the polysystem $\{A_i\}_{i=0,1}$ and of the observation f at q.

6.2 Polysystems and nonlinear differential equation

6.2.1 Nonlinear differential equation (with three singularities)

Let us consider the following singular inputs[33]

$$u_0(z) := z^{-1} \quad \text{and} \quad u_1(z) := (1-z)^{-1}, \tag{118}$$

and the following nonlinear dynamical system[34]

$$\left\{\begin{array}{rcl} y(z) & = & f(q(z)), \\ \dot{q}(z) & = & A_0(q)\ u_0(z) + A_1(q)\ u_1(z), \\ q(z_0) & = & q_0, \end{array}\right. \tag{119}$$

[33] These singular inputs are not included in the studies of Fliess motivated, in particular, by the renormalization of $y(z)$ at $+\infty$ [19, 20].

[34] Any differential equation with singularities in $\{a,b,c\}$, via homographic transformation $(z-a)(c-b)(z-b)^{-1}(c-a)^{-1}$, can be changed into a differential equation with singularities in $\{0,1,+\infty\}$ (the singularities of homographic transformations belonging to the group $\mathcal{G}$).

where, the state $q = (q_1, \ldots, q_n)$ belongs to the complex analytic manifold of dimension n, q_0 is the initial state, the observation f belongs to $\mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!]$ and $\{A_i\}_{i=0,1}$ is the polysystem defined on (114).

Definition 21 ([27]). *The following power series is called* transport operator *of the polysystem $\{A_i\}_{i=0,1}$ and of the observation f*

$$\mathcal{T} \quad := \quad \sum_{w \in X^*} \alpha_{z_0}^z(w)\ \mathcal{A}(w).$$

By the factorization of the monoid by Lyndon words, we have [27]

$$\mathcal{T} = (\alpha_{z_0}^z \otimes \mathcal{A}) \left(\sum_{w \in X^*} w \otimes w \right) = \prod_{l \in \mathcal{L}ynX} \exp[\alpha_{z_0}^z(S_l)\ \mathcal{A}(\check{S}_l)]. \tag{120}$$

Let us consider again the Chen generating series $S_{z_0 \rightsquigarrow z}$ given in (51) of the diffferential forms involed in (DE) of Example 1, *i.e.* $\omega_0(z) = u_0(z)\ dz$ and $\omega_1(z) = u_1(z)\ dz$, verifying the upper bound conditions given on (56).

6.2.2 Asymptotic behaviour of the successive differentiation of the output via extended Fliess fundamental formula

Theorem 23 ([39]). *The Fliess fundamental formula can be extended as follows*

$$y(z) = \mathcal{T} \circ f_{|q_0} = \sum_{w \in X^*} \langle S_{z_0 \rightsquigarrow z} \mid w \rangle \langle \mathcal{A}(w) \circ f_{|q_0} \mid w \rangle = \langle \sigma f_{|q_0} \parallel S_{z_0 \rightsquigarrow z} \rangle.$$

By the factorization of the Lie exponential series L, it follows the expansions of the output y of nonlinear dynamical system with singular inputs,

Corollary 17 ([39]).

$$\begin{aligned}
y(z) \quad &= \quad \sum_{w \in X^*} g_w(z)\ \mathcal{A}(w) \circ f_{|q_0}, \\
&= \quad \sum_{k \geq 0} \sum_{n_1, \ldots, n_k \geq 0} g_{x_0^{n_1} x_1 \ldots x_0^{n_k} x_1}(z)\ \ \mathrm{ad}_{A_0}^{n_1} A_1 \ldots \mathrm{ad}_{A_0}^{n_k} A_1 e^{\log z A_0} \circ f_{|q_0}, \\
&= \quad \prod_{l \in \mathcal{L}ynX} \exp\Big(g_{S_l}(z)\ \mathcal{A}(\check{S}_l) \circ f_{|q_0} \Big), \\
&= \quad \exp\Big(\sum_{w \in X^*} g_w(z)\ \mathcal{A}(\pi_1(w)) \circ f_{|q_0} \Big),
\end{aligned}$$

where, for any word w in X^*, g_w belongs to the polylogarithm algebra.

Since $S_{z_0 \rightsquigarrow z} = \mathrm{L}(z)\mathrm{L}(z_0)^{-1}$ and since $\sigma f_{|_{q_0}}$ and $\mathrm{L}(z_0)^{-1}$ are invariant by $\partial = d/dz$ then $\partial^l y(z) = \langle \sigma f_{|_{q_0}} \parallel \partial^l S_{z_0 \rightsquigarrow z} \rangle = \langle \sigma f_{|_{q_0}} \parallel \partial^l \mathrm{L}(z)\mathrm{L}(z_0)^{-1} \rangle$, for $l \geq 0$.

With the notations of Proposition 3, we get

$$\partial^l y(z) = \langle \sigma f_{|_{q_0}} \parallel [P_l(z)\mathrm{L}(z)]\mathrm{L}(z_0)^{-1} \rangle = \langle \sigma f_{|_{q_0}} \triangleright P_l(z) \parallel \mathrm{L}(z)\mathrm{L}(z_0)^{-1} \rangle. \tag{121}$$

For $z_0 = \varepsilon \to 0^+$, the asymptotic behaviour and the renormalization at $z = 1$ of $\partial^l y(z)$ (or the asymptotic expansion and the renormalization of its Taylor coefficients at $+\infty$) are deduced from Proposition 5 and extend a little bit the results of [39] as follows

Corollary 18. *For any integer l,we have*

$$\begin{aligned}
\partial^l y(1) \quad &\underset{\varepsilon\to 0^+}{\widetilde{}} \quad \langle \sigma f_{|_{q_0}} \triangleright P_l(1-\varepsilon) \parallel e^{-x_1 \log\varepsilon}\, Z_{\sqcup\!\sqcup}\, e^{-x_0\log\varepsilon}\rangle \\
&= \quad \sum_{w\in X^*} \langle \mathcal{A}(w) \circ f_{|_{q_0}} \mid w\rangle \langle P_l(1-\varepsilon)\, e^{-x_1 \log\varepsilon}\, Z_{\sqcup\!\sqcup}\, e^{-x_0\log\varepsilon} \mid w\rangle.
\end{aligned}$$

Corollary 19. *The differentiation of order $l \in \mathbb{N}$ of the output y of the dynamical system (119) is a $\mathcal{C}$-combination of the elements g belonging to the polylogarithm algebra. If its ordinary Taylor expansion exists then the coefficients of this expansion belong to the algebra of harmonic sums and there exists algorithmically computable coefficients $a_i \in \mathbb{Z}, b_i \in \mathbb{N}$ and c_i belong to the $\mathbb{C}$-algrebra generated by $\mathcal{Z}$ and by the Euler's γ constant, such that*

$$\partial^l y(z) = \sum_{n\geq 0} y_n^{(l)} z^n, \qquad y_n^{(l)} \underset{n\to\infty}{\widetilde{}} \sum_{i\geq 0} c_i n^{a_i} \log^{b_i} n.$$

6.3 Differential realization

6.3.1 Differential realization

Definition 22. *The* Lie rank *of a formal power series $S \in \mathbb{K}\langle\!\langle X\rangle\!\rangle$ is the dimension of the vector space generated by $\{S \triangleright \Pi \mid \Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X\rangle\}$, or by $\{\Pi \triangleleft S \mid \Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X\rangle\}$.*

Definition 23. *Let $S \in \mathbb{K}\langle\!\langle X\rangle\!\rangle$ and let us put*

$$\begin{aligned}
\mathrm{Ann}(S) \quad &:= \quad \{\Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X\rangle \mid S \triangleright \Pi = 0\}, \\
\mathrm{Ann}^{\perp}(S) \quad &:= \quad \{Q \in (\mathbb{K}\langle\!\langle X\rangle\!\rangle, \sqcup\!\sqcup) \mid Q \triangleright \mathrm{Ann}(S) = 0\}.
\end{aligned}$$

It is immediate that $\mathrm{Ann}^{\perp}(S) \ni S$ and it follows that (see [20, 46])

Lemma 31. *Let $S \in \mathbb{K}\langle\!\langle X\rangle\!\rangle$. If S is of finite Lie rank, d, then the dimension of* $\mathrm{Ann}^{\perp}(S)$ *equals d.*

By Lemma 3, the residuals are derivations for shuffle product. Then,

Lemma 32. *Let $S \in \mathbb{K}\langle\!\langle X\rangle\!\rangle$. Then :*

1. *For any Q_1 and $Q_2 \in \mathrm{Ann}^{\perp}(S)$, one has $Q_1 \sqcup\!\sqcup\, Q_2 \in \mathrm{Ann}^{\perp}(S)$.*
2. *For any $P \in \mathbb{K}\langle X\rangle$ and $Q_1 \in \mathrm{Ann}^{\perp}(S)$, one has $P \triangleleft Q_1 \in \mathrm{Ann}^{\perp}(S)$.*

Definition 24 ([20])**.** *The formal power series $S \in \mathbb{K}\langle\!\langle X\rangle\!\rangle$ is* differentially produced *if there exists*

- *an integer d,*
- *a power series $f \in \mathbb{K}[\![\bar{q}_1, \ldots, \bar{q}_d]\!]$,*
- *a homomorphism $\mathcal{A}$ from X^* maps to the algebra of differential operators generated by*

$$\mathcal{A}(x_i) = \sum_{j=1}^{d} A_i^j(\bar{q}_1, \ldots, \bar{q}_d)\frac{\partial}{\partial \bar{q}_j}, \quad \mathrm{A}_i^j(\bar{q}_1, \ldots, \bar{q}_d) \in \mathbb{K}[\![\bar{q}_1, \ldots, \bar{q}_d]\!], j = 1, \ldots, d,$$

such that, for any $w \in X^*$, $\langle S \mid w\rangle = \mathcal{A}(w) \circ f_{|0}$.

The couple $(\mathcal{A}, f)$ is called differential representation *of S of dimension d.*

Proposition 24 ([46])**.** *Let $S \in \mathbb{K}\langle\!\langle X\rangle\!\rangle$. If S is differentially produced then it verifies the growth condition and its Lie rank is finite.*

Proof. Let $(\mathcal{A}, f)$ be a differential representation of S of dimension d. Then, by the notations of Definition 20, we get

$$\sigma f_{|0} = S = \sum_{w \in X^*} (\mathcal{A}(w) \circ f)_{|0}\, w.$$

For any $j = 1, .., d$, we put

$$\begin{aligned} T_j &= \sum_{w \in X^*} \frac{\partial(\mathcal{A}(w) \circ f)}{\partial \bar{q}_j}\, w \\ \iff \forall w \in X^*, \quad \langle T_j \mid w\rangle &= \frac{\partial(\mathcal{A}(w) \circ f)}{\partial \bar{q}_j}. \end{aligned}$$

Firstly, by Theorem 22, the generating series σf verifies the growth condition. Secondly, for any $\Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X\rangle$ and for any $w \in X^*$, one has

$$\langle \sigma f \triangleright \Pi \mid w\rangle = \langle \sigma f \mid \Pi w\rangle = \mathcal{A}(\Pi w) \circ f = \mathcal{A}(\Pi) \circ (\mathcal{A}(w) \circ f).$$

Since $\mathcal{A}(\Pi)$ is a derivation over $\mathbb{K}[\![\bar{q}_1, \ldots, \bar{q}_d]\!]$:

$$\begin{aligned} \mathcal{A}(\Pi) &= \sum_{j=1}^{d} (\mathcal{A}(\Pi) \circ \bar{q}_j)\frac{\partial}{\partial \bar{q}_j}, \\ \Rightarrow \quad \mathcal{A}(\Pi) \circ (\mathcal{A}(w) \circ f) &= \sum_{j=1}^{d} (\mathcal{A}(\Pi) \circ \bar{q}_j)\frac{\partial(\mathcal{A}(w) \circ f)}{\partial \bar{q}_j} \end{aligned}$$

then we deduce that

$$\begin{aligned} \forall w \in X^*, \quad \langle \sigma f \triangleright \Pi \mid w\rangle &= \sum_{j=1}^{d} (\mathcal{A}(\Pi) \circ \bar{q}_j)\langle T_j \mid w\rangle, \\ \iff \quad \sigma f \triangleright \Pi &= \sum_{j=1}^{d} (\mathcal{A}(\Pi) \circ \bar{q}_j)\, T_j \end{aligned}$$

That means $\sigma f \triangleright \Pi$ is $\mathbb{K}$-linear combination of $\{T_j\}_{j=1,..,d}$ and the dimension of the vector space $\mathrm{span}\{\sigma f \triangleright \Pi \mid \Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X\rangle\}$ is less than or equal to d. □

6.3.2 Fliess' local realization theorem

Proposition 25 ([46])**.** *Let $S \in \mathbb{K}\langle\langle X\rangle\rangle$ such that its Lie rank equals d. Then there exists a basis $S_1, \ldots, S_d \in \mathbb{K}\langle\langle X\rangle\rangle$ of $(\mathrm{Ann}^{\perp}(S), ⧢) \cong (\mathbb{K}[\![S_1, \ldots, S_d]\!], ⧢)$ such that the S_i's are proper and for any $R \in \mathrm{Ann}^{\perp}(S)$, one has*

$$R = \sum_{i_1,\ldots,i_d \geq 0} \frac{r_{i_1,\ldots,i_n}}{i_1! \ldots i_d!} S_1^{⧢ i_1} ⧢ \ldots ⧢ S_d^{⧢ i_d},$$

where the coefficients $\{r_{i_1,\ldots,i_d}\}_{i_1,\ldots,i_d \geq 0}$ belong to $\mathbb{K}$ and $r_{0,\ldots,0} = \langle R \mid \epsilon\rangle$.

Proof. By Lemma 31, a such basis exists. More precisely, since the Lie rank of S is d then there exists $P_1, \ldots, P_d \in \mathcal{L}ie_{\mathbb{K}}\langle X\rangle$ such that $S \triangleright P_1, \ldots, S \triangleright P_d \in (\mathbb{K}\langle\langle X\rangle\rangle, ⧢)$ are $\mathbb{K}$-linearly independent. By duality, their exists $S_1, \ldots, S_d \in (\mathbb{K}\langle\langle X\rangle\rangle, ⧢)$ such that

$$\forall i, j = 1, ..d, \quad \langle S_i \mid P_j\rangle = \delta_{i,j}, \quad \text{and} \quad R = \prod_{i=1}^{d} \exp(S_i\ P_i).$$

Expending this product, one obtains, via Poincaré-Birkhoff-Witt theorem, the expected expression for the coefficients $r_{i_1,\ldots,i_d} = \langle R \mid P_1^{i_1} \ldots P_d^{i_d}\rangle$. Hence, $(\mathrm{Ann}^{\perp}(S), ⧢)$ is generated by $S_1, \ldots, S_d$. □

With the notations of Proposition 25, one has respectively

Corollary 20. *If $S \in \mathbb{K}[S_1, \ldots, S_d]$ then, for any $i = 0, 1$ and $j = 1, .., d$, one has $x_i \triangleleft S \in \mathrm{Ann}^{\perp}(S) = \mathbb{K}[S_1, \ldots, S_d]$.*

Corollary 21. *The power series S verifies the growth condition if and only if, for any $i = 1, ..d$, S_i also verifies the growth condition.*

Proof. Assume their exists $j \in [1, .., d]$ such that S_j does not verify the growth condition. Since $S \in \mathrm{Ann}^{\perp}(S)$ then using the decomposition of S on $S_1, \ldots, S_d$, one obtains a contradiction with the fact that S verifies the growth condition.

Conservely, using Proposition 20, we get the expected results. □

Theorem 24 ([20])**.** *The formal power series $S \in \mathbb{K}\langle\langle X\rangle\rangle$ is differentially produced if and only if its Lie rank is finite and if it verifies the χ-growth condition.*

Proof. By Proposition 24, one gets a direct proof.

Conversely, since the Lie rank of S equals d then by Proposition 25, by putting $\sigma f_{|_0} = S$ and, for any $j = 1, .., d$, $\sigma\bar{q}_i = S_i$,

1. we choose the observation f as follows

$$f(\bar{q}_1, \ldots, \bar{q}_d) = \sum_{i_1,\ldots,i_d \geq 0} \frac{r_{i_1,\ldots,i_n}}{i_1! \ldots i_d!} \bar{q}_1^{i_1} \ldots \bar{q}_d^{i_d} \quad \in \quad \mathbb{K}[\![\bar{q}_1, \ldots, \bar{q}_d]\!],$$

such that

$$\sigma f_{|_0}(\bar{q}_1, \ldots, \bar{q}_d) = \sum_{i_1,\ldots,i_d \geq 0} \frac{r_{i_1,\ldots,i_n}}{i_1! \ldots i_d!} (\sigma\bar{q}_1)^{⧢ i_1} ⧢ \ldots ⧢ (\sigma\bar{q}_d)^{⧢ i_d},$$

2. it follows that, for $i = 0, 1$ and for $j = 1, .., d$, the residuals $x_i \triangleleft \sigma\bar{q}_j$ belongs to $\mathrm{Ann}^{\perp}(\sigma f_{|0})$ (see also Lemma 32),

3. since σf verifies the χ-growth condition then, by Corollary 21, the generating series $\sigma\bar{q}_j$ and $x_i \triangleleft \sigma\bar{q}_j$ (for $i = 0, 1$ and for $j = 1, .., d$) verify also the growth condition. We then take (see Lemma 29)

$$\forall i = 0, 1, \quad \forall j = 1, .., d, \quad \sigma A^i_j(\bar{q}_1, \ldots, \bar{q}_d) = x_i \triangleleft \sigma\bar{q}_j,$$

   by expressing σA^i_j on the basis $\{\sigma\bar{q}_i\}_{i=1,..,d}$ of $\mathrm{Ann}^{\perp}(\sigma f_{|0})$,

4. the homomorphism $\mathcal{A}$ is then determined as follows

$$\forall i = 0, 1, \quad \mathcal{A}(x_i) \quad = \quad \sum_{j=0}^{d} A^i_j(\bar{q}_1, \ldots, \bar{q}_d)\frac{\partial}{\partial\bar{q}_j},$$

   where, for $i = 0, 1, j = 1, .., d, A^i_j(\bar{q}_1, \ldots, \bar{q}_d) = \mathcal{A}(x_i) \circ \bar{q}_j$ (see Lemma 26).

Thus, $(\mathcal{A}, f)$ provides a differential representation[35] of dimension d of S. □

Moreover, one also has the following

Theorem 25 ([20]). *Let $S \in \mathbb{K}\langle\langle X\rangle\rangle$ supposed to be a differentially produced formal power series. If $(\mathcal{A}, f)$ and $(\mathcal{A}', f')$ are two differential representations of dimension n of S then there exists a continuous and convergent automorphism h of $\mathbb{K}$ such that, for $w \in X^*, g \in \mathbb{K}, h(\mathcal{A}(w)\circ g) = \mathcal{A}'(w)\circ(h(g))$ and $f' = h(f)$.*

Since any rational power series (resp. polynomial), verifies the growth condition and its Lie rank is less or equal to its Hankel rank which is finite [20] then

Corollary 22. *Any rational power series and any polynomial over X with coefficients in $\mathbb{K}$ are differentially produced.*

Remark 5. *1. By Corollary 20, if S is polynomial then for any $j = 1, .., d$, S_j is polynomial. Therefore, for $i = 0, 1$ and $j = 1, .., d$, $x_i \triangleleft S$ is also polynomial over X. In this case, let $(\mathcal{A}, f)$ be a differential representation of S of dimension d. Then f and $\{A^i_j\}^{i=0,1}_{j=1,..,d}$ are obviously polynomial on $\bar{q}_1, \ldots, \bar{q}_d$ and the Lie algebra generated by $\{\mathcal{A}(x_i)\}^{i=0,1}$ is nilpotent.*

2. Note also that, by Theorem 6, if S is rational over X of linear representation (λ, μ, η) then the observation $f(q_1, \ldots, q_n)$ equals $\lambda_1 q_1 + \ldots + \lambda_n q_n$ and the polysystem $\{\mathcal{A}(x)\}_{x\in X}$ is obtained by putting

$$\forall x_i \in X, \quad \mathcal{A}(x_i) \quad = \quad \sum_{j=1}^{n} (\mu(x_i))^i_j \frac{\partial}{\partial q_j}$$

yields linear *representation not necessarily of minimal dimension [20].*

[35] In [20, 46], the reader can found the discussion on the *minimal* differential representation.

3. Assume $S \in \mathbb{K}\epsilon \oplus x_0\mathbb{K}\langle\langle X\rangle\rangle x_1$ and S is a differentially produced. If there exists a basis $S_1, \ldots, S_d$ of $(\mathrm{Ann}^{\perp}(S), ш)$ $\cong (x_0\mathbb{K}\langle\langle X\rangle\rangle x_1, ш)$ *such that*

$$S = \sum_{i_1,\ldots,i_d \geq 0} r_{i_1,\ldots,i_n} \frac{S_1^{ш\, i_1}}{i_1!} ш \ldots ш \frac{S_d^{ш\, i_d}}{i_d!} \in \ (\mathbb{K}[S_1, \ldots, S_d], ш). \quad (122)$$

We put $\Sigma_i := \pi_Y S_i$, for $i = 1, .., d$ and then

$$\Sigma := \sum_{i_1,\ldots,i_d \geq 0} r_{i_1,\ldots,i_n} \frac{\pi_1^{ш\!\!\!+ i_1}}{i_1!} ш\!\!\!+ \ldots ш\!\!\!+ \frac{\Sigma_d^{ш\!\!\!+ i_d}}{i_d!} \in (\mathbb{K}[\Sigma_1, \ldots, \Sigma_d], ш\!\!\!+). \ (123)$$

It is a generalization of a Radford's theorem because [29, 30] :

- *If $S \in \mathbb{Q}\langle X\rangle$ then (122), (123) are decompositions on Radford bases.*
- *If S is rational then these are* noncommutative partial decompositions*. In general one has $\pi_Y S \neq \Sigma$ but $\zeta(S_i) = \zeta(\Sigma_i)$ and*

$$\zeta(S) = \zeta(\Sigma) = \sum_{i_1,\ldots,i_d \geq 0} r_{i_1,\ldots,i_n} \frac{\zeta(S_1)^{i_1}}{i_1!} \ldots \frac{\zeta(S_d)^{i_d}}{i_d!}. \quad (124)$$

Thus, these yield also identities on polyzêtas at arbitrary weight [37].

References

[1] E. Abe.– Hopf algebra, Cambridge, 1980.

[2] J. Berstel, and C. Reutenauer.– Rational series and their languages, Springer-Verlag, 1988.

[3] M. Bigotte.– Etude symbolique et algorithmique des fonctions polylogarithmes et des nombres dEuler-Zagier colorés. Thèse, Université Lille, 2000.

[4] L. Boutet de Monvel.– Remark on divergent multizeta series, in Microlocal Analysis and Asymptotic Analysis, RIMS workshop 1397 (2004), pp. 1-9.

[5] N. Bourbaki.– Fonctions of Real Variable, Springer.

[6] N. Bourbaki.– Algebra, chapters II et III, Springer.

[7] P. Cartier.– Développements récents sur les groupes de tresses. Applications à la topologie et à l'algèbre, Sém BOURBAKI, 42$^{\text{ème}}$ 1989-1990, n°716.

[8] P. Cartier.– Fonctions polylogarithmes, nombres polyzêtas et groupes prounipotents, Sém BOURBAKI, 53$^{\text{ème}}$ 2000-2001, n°885.

[9] R. Chari & A. Pressley.– A guide to quantum group, Cambridge, (1994).

[10] K.T. Chen.– *Iterated path integrals*, Bull. Amer. Math. Soc., vol 83, 1977, pp. 831-879.

[11] G. Duchamp & C. Reutenauer.– Un critère de rationalité provenant de la géométrie noncommutative, Invent. Math., pp. 613-622, (1997).

[12] Costermans, C., Enjalbert, J.Y. and Hoang Ngoc Minh.– Algorithmic and combinatoric aspects of multiple harmonic sums, *Discrete Mathematics & Theoretical Computer Science Proceedings*, 2005.

[13] G. H. E. Duchamp, C. Tollu.– Sweedler's duals and Schützenberger's calculus, "Conference on Combinatorics and Physics", `arXiv:` 0712.0125v3.

[14] V. Drinfel'd.– Quantum group, Proc. Int. Cong. Math., Berkeley, 1986.

[15] V. Drinfel'd.– Quasi-Hopf Algebras, *Len. Math. J.*, 1, 1419-1457, 1990.

[16] V. Drinfel'd.– On quasitriangular quasi-hopf algebra and a group closely connected with gal$(\bar{q}/q)$, *Leningrad Math. J.*, 4, 829-860, 1991.

[17] J. Ecalle.– ARI/GARI, la dimorphie et l'arithmétique des multizêtas : un premier bilan, J. Th. des nombres de Bordeaux, 15, (2003), pp. 411-478.

[18] Fliess M.– Matrices de Hankel, J. M. Purs. Appl., 53, pp. 197-222, (1974).

[19] Fliess M.– *Fonctionnelles causales non linéaires et indéterminées non commutatives*, Bull. SMF, N°109, pp. 3-40, (1981).

[20] Fliess M.– *Réalisation locale des systèmes non linéaires, algèbres de Lie filtrées transitives et séries génératrices non commutatives*, Inventiones Mathematicae, Volume 71, Number 3 / mars 1983, pp. 521-537.

[21] K. Ihara, M. Kaneko & D. Zagier.– Derivation and double shuffle relations for multiple zetas values, Compositio Math. 142 (2006), pp. 307-338.

[22] T.Q.T. Lê & J. Murakami.– Kontsevich's integral for Kauffman polynomial, Nagoya Math., pp 39-65, 1996.

[23] C. Malvenuto & C. Reutenauer.– Duality between quasi-symmetric functions and the solomon descent algebra, J. of Alg. 177 (1995), pp. 967-982.

[24] C. Hespel.– Une étude des séries formelles noncommutatives pour l'Approximation et l'Identification des systèmes dynamiques, thèse docteur d'état, Université Lille 1, (1998).

[25] Hoang Ngoc Minh.– Contribution au développement d'outils informatiques pour résoudre des problèmes d'automatique non linéaire, Thèse, Lille, 1990.

[26] Hoang Ngoc Minh.– *Input/Output behaviour of nonlinear control systems : about exact and approximated computations*, IMACS-IFAC Symposium, Lille, Mai 1991.

[27] Hoang Ngoc Minh, G. Jacob, N. Oussous.– *Input/Output Behaviour of Nonlinear Control Systems : Rational Approximations, Nilpotent structural Approximations*, in *Analysis of controlled Dynamical Systems*, (B. Bonnard, B. Bride, J.P. Gauthier & I. Kupka eds.), Progress in Systems and Control Theory, Birkhäuser, 1991, pp. 253-262.

[28] Hoang Ngoc Minh.– *Summations of Polylogarithms via Evaluation Transform*, dans Math. & Computers in Simulations, 1336, pp 707-728, 1996.

[29] Hoang Ngoc Minh.– Fonctions de Dirichlet d'ordre n et de paramètre t, dans Discrete Mathematics 180, pp 221-242, 1998.

[30] Hoang Ngoc Minh.– *Calcul symbolique non commutatif : aspects combinatoires des fonctions spéciales et des nombres spéciaux*, HDR, Lille 2000.

[31] Hoang Ngoc Minh, G. Jacob.– Symbolic integration of meromorphic differential systems via Dirichlet functions, Discrete Mathematics 210 (2000), pp 87-116.

[32] Hoang Ngoc Minh, Jacob G., N.E. Oussous, M. Petitot.– Aspects combinatoires des polylogarithmes et des sommes d'Euler-Zagier, *journal électronique du Séminaire Lotharingien de Combinatoire*, B43e, (2000).

[33] Hoang Ngoc Minh, Jacob G., N.E. Oussous, M. Petitot.– De l'algèbre des ζ de Riemann multivariées l'algèbre des ζ de Hurwitz multivariées, *journal électronique du Séminaire Lotharingien de Combinatoire*, 44, (2001).

[34] Hoang Ngoc Minh & M. Petitot.– Lyndon words, polylogarithmic functions and the Riemann ζ function, Discrete Math., 217, 2000, pp. 273-292.

[35] Hoang Ngoc Minh, M. Petitot and J. Van der Hoeven.– Polylogarithms and Shuffle Algebra, *Proceedings of FPSAC'98*, 1998.

[36] Hoang Ngoc Minh, Petitot, M., and Van der Hoeven, J..– L'algèbre des polylogarithmes par les séries génératrices, *Proceedings of FPSAC'99*, 1999.

[37] Hoang Ngoc Minh.– Finite polyzêtas, Poly-Bernoulli numbers, identities of polyzêtas and noncommutative rational power series, *Proceedings of* 4^{th} *International Conference on Words*, pp. 232-250, 2003.

[38] Hoang Ngoc Minh.– Differential Galois groups and noncommutative generating series of polylogarithms, in "Automata, Combinatorics and Geometry", 7th World Multi-conference on Systemics, Cybernetics and Informatics, Florida, 2003.

[39] Hoang Ngoc Minh.– Algebraic Combinatoric Aspects of Asymptotic Analysis of Nonlinear Dynamical System with Singular Inputs, Acta Academiae Aboensis, Ser. B, Vol. 67, no. 2, (2007), pp. 117-126.

[40] M. Hoffman.– The Multiple harmonic series, Pacific Journal of Mathematics, 152, 2 (1992), pp. 275-290.

[41] M. Hoffman.– The algebra of multiple harmonic series, J. of Alg., (1997).

[42] G. Hochschild.– The structure of Lie groups, Holden-Day, (1965).

[43] G. Racinet.– Doubles mélanges des polylogarithmes multiples aux racines de l'unité, Publications Mathématiques de l'IHÉS, 95 (2002), pp. 185-231.

[44] R. Ree.– *Lie elements and an algebra associated with shuffles*, Ann. of Math, 68 (1958), 210–220.

[45] C. Reutenauer.– Free Lie Algebras, London Math. Soc. Monographs, 1993.

[46] C. Reutenauer.– *The local realisation of generating series of finite Lie rank.* Algebraic and Geometric Methods In Nonlinear Control Theory. 33-43.

[47] M.P. Schützenberger.– On the definition of a family of automata, Information and Control, 4 (1961) 245-270.

[48] M. Waldschmidt.– Hopf Algebra and Transcendental numbers, "Zeta-functions, Topology and Quantum Physics 2003", Kinki : Japan (2003).

[49] El Wardi.– Mémoire de DEA, Université de Lille, 1999.

[50] D. Zagier.– Values of zeta functions and their applications, in "First European Congress of Mathematics", vol. 2, Birkhäuser (1994), pp. 497-512.